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Ranking Sets of Arguments in Abstract Argumentation

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RANKING SETS OF ARGUMENTS IN ABSTRACT ARGUMENTATION

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ABSTRACT

Argumentation is a crucial process for rational decision-making, as it involves resolving conflicts between opposing opinions to find acceptable outcomes. The field of *Computational Models of Argumentation* addresses the reasoning process using arguments and the attacks between them to model an *abstract argumentation framework* as a directed graph. Traditionally, reasoning with these frameworks involves binary classification, where a set of arguments is either accepted or rejected. But, such binary classification is not always sufficient.

In this thesis, we explore various approaches to rank sets of arguments within an abstract argumentation framework, providing more nuanced reasoning that goes beyond traditional binary classification. These approaches allow us to determine not only whether sets of arguments are accepted but also whether one set is more "acceptable" than another. We introduce the notion of *extension-ranking semantics*, which creates a preorder over the powerset of arguments, enabling us to compare the plausibility of acceptance of different sets. One family of extension-ranking semantics are generalisations of Dung's extension semantics, maintaining the binary classification between acceptable and rejected sets while also allowing comparisons between rejected sets. This enables us to determine if one rejected set is "closer" to acceptance than another.

We leverage the parallels between formal argumentation and *conditional logic* to define an extension-ranking semantics. Conditional logic studies statements of the form "if A then B". In this context, an attack between two arguments, a and b, in an abstract argumentation framework can be interpreted as a conditional of the form "if argument a is accepted, then argument b is not accepted". Using these conditional, we can rank sets of arguments according to their plausibility of acceptance.

For the final family of extension-ranking semantics, we utilise *argument-ranking semantics*, which rank individual arguments within an abstract argumentation framework based on their strength. We then transform these rankings over arguments into a ranking over sets of arguments, where sets containing highly ranked arguments are themselves highly ranked.

Beside presenting extension-ranking semantics, we analyses them axiomatically and in terms of computational complexity. We identify principles that a well-behaved extension-ranking semantics should satisfy, modelling key aspects of argumentative reasoning. We also investigate the computational complexity of comparing two sets of arguments.

As a final contribution, we explore *social ranking functions* within the context of abstract argumentation. The social ranking problem involves transforming a ranking over sets of objects into a ranking over individual objects. We examines the relationship between extension rankings and argument rankings, as well as the connection between social ranking functions and argumentation.

PUBLICATIONS AND DISCLAIMER

This thesis builds upon ideas and content from several previously published papers. Consequently, some passages in this thesis may resemble those in earlier publications. Scientific research is a collaborative effort, involving discussions and collaborations with other researchers. We acknowledge the contributions of various researchers alongside the author of this thesis.

Much of the work presented in this thesis was supported by the DFG-project Conditional Argumentative Reasoning (CAR)[1] in collaboration with Gabriele Kern-Isberner, Jesse Heyninck, Matthias Thimm, and Tjitze Rienstra (Skiba and Thimm, 2020; Heyninck et al., 2021a; Skiba et al., 2021b; Heyninck et al., 2021b, 2022a; Skiba and Thimm, 2022; Skiba et al., 2022; Rienstra et al., 2022; Heyninck et al., 2022b, 2023, 2024). Some of these papers are integral to this thesis, while others have influenced it.

The publications (Bengel et al., 2025, 2024a) , particularly the proofs, are based on extensive discussions between Lars Bengel, Giovanni Buraglio, Jan Maly, and the author of this thesis.

The author has also contributed to other publications not included in this thesis. The papers (Skiba et al., 2020a, 2021a; Neugebauer et al., 2021) investigate the problem of existence of non-empty extensions in argumentation frameworks with (qualified) uncertainty.

In the paper (Skiba and Thimm, 2024), two soft notions for admissibility to optimise and approximate in abstract argumentation are discussed.

In another publication the problem of *distinguishability* of extension semantics in abstract argumentation frameworks are addressed (Kuhlmann et al., 2021).

The papers (Skiba et al., 2023, 2024) discuss and analyse argument-ranking semantics for *assumption-based argumentation*.

The paper (Skiba et al., 2020b) introduces an extension of abstract argumentation frameworks to incorporate evidence for arguments. Each piece of evidence has its own cost of retrieval. The goal is to decide on the acceptance of arguments with minimal cost.

Beyond formal argumentation, the author has worked in the area of blockchains, investigating *cost fairness* in exchange protocols (Lohr et al., 2022a,b).

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Finally, to my friends and family – a huge thank you for always supporting me and for providing much-needed distractions from the stress of writing. A special thank you to my parents for encouraging me to continue with my studies and always believing in me.

CONTENTS

INTRODUCTION 1

Rational decision-making is one of the key tasks of *Artificial Intelligence* (AI) (Russell and Norvig, 2020). Decisions are based on information derived from data, which is not flawless and may encompass contradictions. Conflicting information is prevalent across various fields, including law (Bench-Capon, 1997; Bench-Capon et al., 2009; Bench-Capon, 2020; Prakken, 1995; Prakken and Sartor, 1996; Prakken, 2006; Prakken and Sartor, 2015), medicine (Craven et al., 2012; Fox et al., 2007; Hunter and Williams, 2012; Tolchinsky et al., 2006, 2012) or robotics (Kam et al., 1997; Novak and Riener, 2015). To illustrate, consider a scenario from the field of law. John is the prime suspect in a murder case. Despite possessing an alibi, a witness, Jack, claims to have seen John at the crime scene. Consequently, the judge is presented with two contradictory statements: "John was at the scene of the crime" and "John was not at the scene of the crime". The rational judge must decide between these conflicting pieces of information, as accepting both simultaneously is not possible.

To resolve conflicts and facilitate rational decision-making, humans engage in *argumentation*. This process involves presenting arguments for or against specific statements to reach a conclusion. In the given example, John possesses an alibi, which is an argument against his guilt ("John has an alibi, so he cannot be the murderer"). Conversely, Jack's testimony is an argument against John's alibi ("Because John was seen at the scene of the crime, John does not have an alibi"). However, during the court hearing, it is revealed that Jack was intoxicated when he allegedly saw John, leading the judge to question the reliability of Jack's testimony. The judge identifies an argument against Jack's testimony ("Jack is unreliable"). Finally, based on the arguments presented, the judge determines whether John is guilty. The example demonstrates the process of argumentation. First a statement is presented for discussion ("John is guilty"), followed by the presentation of arguments for or against this statement ("John has an alibi"). Subsequently, arguments against these arguments are presented ("John was seen that the scene of the crime"). This iterative process continues until no further arguments are presented. Finally, a set of arguments is identified that jointly support or oppose the initial statement, leading to a rational decision. Given that humans employ argumentation to make decisions, studying formal models of argumentation in AI is of considerable interest for modelling rational decision-making processes that more closely align with human reasoning.

Formal argumentation (Atkinson et al., 2017) is a subfield of *knowledge representation and reasoning* (Brachman and Levesque, 2004; van Harmelen et al., 2008; Beierle and Kern-Isberner, 2019), a research field in AI concerned with various methods of representing knowledge and analysing the capacity of these methods to model human-like reasoning. The origins of argument-based reasoning in AI can be traced back to the seminal works of Pollock (1987, 1991, 2001, 1992, 1994, 1995). Approaches to formal argumentation are characterized as *non-monotonic* formalisms (Brewka, 1991), meaning that the addition of new knowledge to the knowledge base can alter previously held beliefs. For instance, in the given example, the presentation of the new argument that Jack was intoxicated modifies the knowledge base, potentially influencing the judge's decision-making process.

The argumentation pipeline consists of several steps (Caminada and Amgoud, 2007). The initial step involves identifying arguments from the available information. Given the vast amount of data, particularly in the era of the World Wide Web and generative AI, a systematic process is essential for computing arguments. Various approaches utilising machine learning and natural language processing have been developed to extract or *mine* arguments from unstructured information (Lippi and Torroni, 2016; Cabrio and Villata, 2018; Lawrence and Reed, 2019). The subsequent step is to identify conflicts among the mined arguments (Walton et al., 2008). Once the conflicts identified, an argumentation framework can be constructed to represent the argumentative process. A prominent approach for this purpose is the *abstract argumentation framework* (AF) proposed by Dung (1995), which represents argumentative scenarios as directed graphs. In this approach, *arguments* are represented as vertices, and an *attack* from one argument to another are depicted as a directed edge. Abstract argumentation frameworks focus solely on the relationships between arguments, disregarding their content. The final step in the argumentation pipeline is to evaluate the *acceptability* of sets of arguments and derive a conclusion. In AFs, this evaluation is usually conducted by considering *extensions*, which are sets of arguments that are jointly acceptable according to some formal account of acceptability. One method of determining the acceptability of a set is to examine the internal conflicts. A set of arguments is deemed "acceptable" if no two arguments within it attack each other. A more refined concept is *admissibility*, which states that a set is only acceptable if it can defend itself against any threats. Acceptability concepts such as admissibility lead to *extension semantics*, which differentiate between "acceptable" and "not acceptable" sets of arguments.

The binary classification inherent in extension semantics proves overly restrictive, particularly when the goal is to compare sets of arguments based on their acceptability. This limitation becomes evident when examining more nuanced scenarios, such as the given law example detailed below (for the formal definitions, see Chapter 2).

Example 1. *John is the prime suspect in a murder case and possesses an alibi for the time of the crime. A witness, Jack, claims having seen John at the crime scene. However, it is later revealed that Jack was intoxicated, rendering his testimony unreliable and*

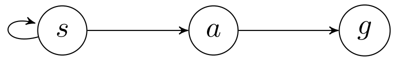

Figure 1.1: AF F_1 from Example 1.

subject to scrutiny. This scenario can be modelled using an abstract argumentation framework $F_1 = (\{g, a, s\}, \{(a, g), (s, a), (s, s)\})$ *where:*

- g *represent the argument* "John is guilty"*,*
- a *represent the argument* "John has an alibi"*,*
- s *represent the argument* "John was seen at the scene of the crime"*.*

In this framework, the argument g is attacked by a, and a is attacked by s. Notably, s is unreliable and therefore attacks itself. The corresponding AF is depicted in Figure 1.1. The judge decides which position is most plausible. John is either guilty *or not, implying that g is either part of an acceptable set or not. When comparing the two positions of $\{a\}$ and $\{g, s\}$, the judge concludes that neither set is acceptable with respect to admissibility. This is because neither set can defend itself against its attackers. However, while the set $\{a\}$ does not defend itself against the attack of s, at least this set has no internal conflicts. In contrast, the set $\{g, s\}$ has an internal conflict due to the self-attacking nature of s.*

In Example 1 above, we observe an order between two non-acceptable sets. The set $\{a\}$ is "closer" to being acceptable than the set $\{g, s\}$. Extending this analysis, we can examine the relationship with the set $\{a, g, s\}$. Both $\{g, s\}$ and $\{a, g, s\}$ share one conflict (the self-attack of s), but $\{a, g, s\}$ has additional conflicts. Consequently, $\{g, s\}$ is considered "closer" to being acceptable than $\{a, g, s\}$, and similarly, the set $\{a\}$ is also "closer" to being acceptable than $\{a, g, s\}$. This establishes an order for these sets based on their plausibility of acceptance. Such an order is more fine-grained than the binary classification of extension semantics.

In the literature (Amgoud and Ben-Naim, 2013; Amgoud et al., 2016; Bonzon et al., 2016; Yun et al., 2018; Grossi and Modgil, 2019) we can find a number of approaches to determine the relative degree to plausibly accept an individual argument. These approaches, known as *argument-ranking*, *graded* or *gradual* semantics, associate each abstract argumentation framework with an ordering of arguments. However, the plausibility of acceptance of each argument cannot be directly be used to infer the plausibility of acceptance of a set of arguments, because these approaches do not facilitate the representation of a set of arguments as "jointly" acceptable. Specifically, two arguments with a high degree of plausibility to be accepted may not be accepted simultaneously if they are in conflict with each other.

The overall objective of this thesis is to introduce notions for ranking sets of arguments according to their plausibility of acceptance within an abstract argumentation framework and to analyse their behaviour.

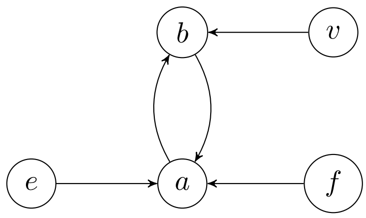

Figure 1.2: AF F_2 for Example 2.

1.1 RESEARCH QUESTIONS

The primary research question of this thesis can be summarised as follows:

"*How can a new notion of graded acceptability be defined?*"

The goal of this work is to provide a more fine-grained evaluation of acceptability than the classical binary classification. Example 1 already illustrates the need for such a nuanced evaluation. To achieve this, we aim to define a reasoning method, we call *extension rankings*, that ranks sets of arguments according to their plausibility of acceptance. Before delving into the details of this concept, we motivate the use of such a ranking further with an example.

Example 2. *Alice and Bob are planning a joint holiday for next year. Living in Germany, they have researched potential destinations online. Alice suggests Berlin as a option, but Bob, who recently visited Berlin for a business trip, is not keen on returning. Instead, Bob proposes Argentina. However, Alice is hesitant about this idea due to her fear of flying and the high cost of flights to Argentina.*

Despite their differing preferences, both Alice and Bob are determined to travel next year. Their options are limited to Berlin and Argentina, since time and financial constraints prevent them from visiting both locations.

To assist Alice and Bob in their decision-making process, we model their problem using an abstract argumentation framework. The two potential destinations, Argentina (a) and Berlin (b), are represented as arguments that are attacked by specific reasons for rejection, i. e. the arguments "expensive" (e) and "fear of flying" (f) attack Argentina while the argument "visited" (v) attacks Berlin. This structure is depicted as F_2 in Figure 1.2, or formally described as an argumentation framework:

$$F_2 = (\{a, b, e, f, v\}, \{(a, b), (b, a), (e, a), (f, a), (v, b)\})$$

Under any extension semantics defined by Dung (1995), the arguments a and b are never part of an acceptable set due to their respective attack. However, given Alice and Bob's constraint that they wish to travel to either Argentina or Berlin (a or b), the task is to determine the most plausible set containing either a or b. We begin by examining the attacks of a and b. Since neither argument can defend itself against its attackers, we temporarily disregard these attacks to explore scenarios where one of these arguments can be accepted. For any set containing a, the attacks from e and f must *to be disregarded, whereas for a set containing b, only the attack from v need to be disregarded. Initially, this suggest that b might be easier to accept. Starting with $\{b\}$, the arguments v, e and f are not attacked, providing no immediate reason for their rejection. However, adding v to the set creates an internal conflict with b, indicating that v should not be included. Conversely, there is no reason to reject e and f, so these arguments should be added, resulting in the final set $\{b, e, f\}$. This set is the most plausible set containing either a or b when considering the method of dismissing attacks, as no additional arguments can be added without conflict. Furthermore, this set is more plausible than any of its subsets containing b.*

In Example 2, the approach does not discredit any of the acceptable sets derived from the abstract argumentation framework F_2. Instead, it enables the comparison of sets of arguments that are not deemed acceptable under Dung's extension semantics, allowing for a nuanced evaluation of which set is more favourable. This observation leads to a research question:

(Q1) *How can we define a reasoning notion in abstract argumentation that is more fine-grained than a binary classification in abstract argumentation?*

In the context of Example 2, a conflict-free set of arguments was identified to meet Alice and Bob's constraints. However, in certain scenarios, constraints or requirements may be impractical or conflicting, necessitating the consideration of conflicting arguments to partially satisfy these constraints. Otherwise, the only remaining option would be to reject all arguments, which may not align with the intended decision-making outcome.

Example 3. *Consider a scenario where a burger restaurant receives the following order: "I want a vegan burger with meat, but please no pork or beef, and please add hot sauce, but do not make the burger spicy." It is important to note that the restaurant exclusively offers pork, beef, or vegan patties. This request can be formalised as an argumentation framework F_3 :*

$$F_3 = (\{v, m, \bar{p}, \bar{b}, s, \bar{s}\}, \{(v, m), (m, v), (m, \bar{p}), (\bar{p}, m), (m, \bar{b}), (\bar{b}, m), (h, \bar{s}), (\bar{s}, h)\})$$

where:

- v *represents a* vegan patty,
- m *represents* extra meat,
- $\bar{p}$ *represents* no pork,
- $\bar{b}$ *represents* no beef,
- h *represents* hot sauce,
- $\bar{s}$ *represents* not spicy.

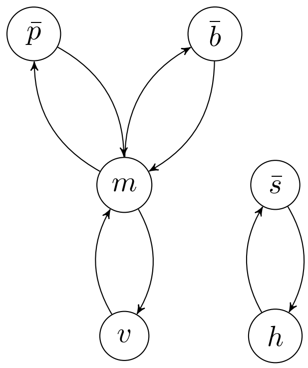

Figure 1.3: AF F_3 for Example 3.

As illustrated in Figure 1.3. The requirements specified in the order are inherently contradictory and cannot be simultaneously satisfied without creating conflicts. Specifically, adding hot sauce (h) *would violated the "not spicy"* $(\bar{s})$ *constraint, and including a meat patty would contradict the vegan* (v) *constraint. Furthermore, the restaurant's meat patty options are limited to pork and beef, both of which the customer wishes to avoid. Given these constraints, the logical course of action would be to refuse the order entirely. However, if the restaurant aims to fulfil the order as closely as possible, certain constraints must be relaxed. Adding a meat patty would violate two constraints (vegan and either no pork or no beef), so the restaurant opts to use a vegan patty. Regarding the hot sauce, either choice violates one constraint. The restaurant randomly decided to not add hot sauce. Consequently, the customer receives a burger with a vegan patty and no hot sauce, as this configuration most closely adheres to the customer's constraints while minimising conflicts.*

In Example 3, we encounter a situation where the given constraints are in conflict with one another and are thus nonsensical. There is no feasible solution that satisfies these constraints. Nevertheless, we aim to present the "closest" or most meaningful solution to this problem. Based on this observation, we formulate the following research question:

(Q2) *How can we define meaningful notion for an decision making process under nonsensical or conflicting constraints in abstract argumentation?*

So far, we have recognised the need to rank sets of arguments but have not yet discussed the criteria for creating these rankings. Our goal is to develop a more nuanced reasoning approach than binary classification while ensuring consistency with established binary classification. Specifically, the highest-ranked set should align with Dung's extension semantics, ensuring acceptability. The overall ranking should be logically consistent, with conflicting sets appropriately penalised and ranked lower. This leads us to the following research question:

(Q3) *What are the characteristics of a well-behaved extension ranking?*

To achieve this research question, we define a set of principles to guide the evaluation and development of notions to rank sets of arguments. These principles should capture key aspects of argumentative reasoning, ensuring that the resulting extension ranking is well-behaved.

Within the field of abstract argumentation, two main reasoning approaches are recognised: extension-based reasoning, as exemplified by Dung's extension semantics, and argument-based reasoning, illustrated by argument-ranking semantics (Amgoud and Ben-Naim, 2013). The relationship between these two approaches has not been extensively explored. However, extension rankings present a potential method for bridging these distinct reasoning approaches, leading to the following research question:

(Q4) *How can extension-based and argument-based reasoning be connected in abstract argumentation?*

We have previously discussed our goal for a principle-based evaluation framework for extension rankings. To fully understand their practical applicability, it is crucial to examine their computational complexity. Specifically, we need to determine how efficiently sets of arguments can be ranked. This leads to the final research question:

(Q5) *What is the computational complexity of computing an extension ranking?*

1.2 PUBLICATION AND CONTRIBUTIONS

In the following, we discuss the publications relevant to this thesis and how they relate to the research questions and this thesis in general.

(Skiba et al., 2021b)	Title:	Ranking Extensions in Abstract Argumentation
	Author:	**K. Skiba**, T. Rienstra, M. Thimm, J. Heyninck, and G. Kern-Isberner
	Venue:	The Thirtieth International Joint Conference on Artificial Intelligence (IJCAI)
	Year:	2021
	Chapters:	3, 4, 5, 6

The paper (Skiba et al., 2021b) serves as the foundational work for this thesis. The primary contribution of this paper is the introduction of *extension-ranking semantics*, which is a formal notion for ranking sets of arguments. This notion enables the comparison of two sets of arguments, E and E', within an abstract argumentation framework, allowing us to state that E is at least as plausible to be accepted as E'.

This paper presents a family of extension-ranking semantics that generalise Dung's extension semantics and proposes several principles for evaluating these semantics. Thus, this paper addresses research questions Q1, Q2, and Q3.

(Skiba et al., 2025)	Title:	Extension-ranking Semantics for Abstract Argumentation Preprint
	Author:	**K. Skiba**, T. Rienstra, M. Thimm, J. Heyninck, and G. Kern-Isberner
	Venue:	arXiv preprint arXiv:2504.21683
	Year:	2025
	Chapters:	3, 4, 5, 6, 8

The paper (Skiba et al., 2025) is an extended version of (Skiba et al., 2021b). It has been submitted to the journal *Journal of Artificial Intelligence Research* and is currently under review. This extended paper includes a more detailed discussion on extension-ranking semantics and introduces additional extension-ranking semantics. The original contribution of this work is presented in Chapter 8, where the combination of argument-ranking semantics and extension-ranking semantics is explored. Therefore, this paper addresses research questions Q1, Q2, Q3, and Q4.

(Skiba and Thimm, 2022)	Title:	Ordinal Conditional Functions for Abstract Argumentation
	Author:	**K. Skiba**, and M. Thimm
	Venue:	The Ninth International Conference on Computational Models of Argument (COMMA)
	Year:	2022
	Chapters:	7

In the publication (Skiba and Thimm, 2022), the parallels between abstract argumentation and *conditional logic* are leveraged to define extension-ranking semantics. Conditional logic is a branch of logic that examines *conditionals*, which are structures of the form "if A then B". The truth values of these conditionals are contingent on the truth values of their components, A and B. In an abstract argumentation framework an attack between two arguments is interpreted as a conditional of the form "if argument a is accepted, then argument b is not accepted". For these types of conditionals, *ordinal conditional functions* are defined to rank sets of arguments. This paper addresses research questions Q1 and Q2 by providing a notion for ranking sets of arguments under conflicting constraints. Portions of this research are also incorporated into the book chapter by Heyninck et al. (2024).

(Skiba et al., 2022)	Title:	Realisability of Rankings-based Semantics
	Author:	**K. Skiba**, T. Rienstra, M. Thimm, J. Heyninck, and G. Kern-Isberner
	Venue:	The Fourth International Workshop on Systems and Algorithms for Formal Argumentation (SAFA)
	Year:	2022
	Chapters:	2, 8

In the paper (Skiba et al., 2022), the realisability question for argument-ranking semantics was investigated. This question addresses whether, given an argument ranking (a preorder over arguments), does there exists an abstract argumentation framework that induces that argument ranking according to an argument-ranking semantics. Under certain conditions, this problem can be trivially answered with `yes`. Furthermore, an equivalence notion for abstract argumentation frameworks with respect to argument-ranking semantics was introduced. It is demonstrated that even if two abstract argumentation frameworks induce the same σ-extensions, they can still induce different argument rankings. Therefore, this paper primarily focuses on research question Q4, exploring the connections between extension-based and argument-based reasoning in abstract argumentation.

(Skiba, 2023)	Title:	Bridging the Gap between Ranking-based Semantics and Extension-ranking Semantics
	Author:	**K. Skiba**
	Venue:	The Ninth Workshop on Formal and Cognitive Reasoning (FCR)
	Year:	2023
	Chapters:	8, 9

The paper (Skiba, 2023) discusses the relationship between argument-ranking semantics and extension-ranking semantics. The primary contributions of this work include the development of transformations from argument-ranking semantics to extension-ranking semantics, referred as *liftings*, and from extension-ranking semantics to argument-ranking semantics, referred as *social rankings*. Additionally, the paper identifies and discusses some of the limitations and shortcomings associated with these transformations. This study primarily addresses research question Q4, focusing on the interplay and integration of these two reasoning approaches within abstract argumentation.

In the paper (Bengel et al., 2025), an analysis of the *social ranking problem* within the context of abstract argumentation is presented. The social ranking problem explores how a ranking over sets of objects can be transformed into a ranking over individual objects. Specifically, the study examines the relationship between extension rankings and argument rankings. The primary contribution of this paper is the introduction of an argument-ranking semantics derived from an extension-ranking

(Bengel et al., 2025)	Title:	An Extension-Based Argument-Ranking Semantics: Social Rankings in Abstract Argumentation
	Author:	L. Bengel, G. Buraglio, J. Maly and **K. Skiba**
	Venue:	Thirty-ninth AAAI Conference on Artificial Intelligence (AAAI)
	Year:	2025
	Chapters:	4, 9

semantics. The principles underlying this transformation are discussed and analysed, revealing that the newly proposed argument-ranking semantics refines the classification of arguments into skeptically accepted, credulously accepted, and rejected arguments. This work primarily addresses research question Q4. The paper is an extended conference version of the workshop paper by Bengel et al. (2024a).

Regarding research question Q5, which focuses on the computational complexity of extension-ranking semantics, the discussion and analysis presented in this thesis are original contributions and have not been previously published. This material is primarily covered in Chapter 10. Additionally, there are other unpublished sections of the thesis, such as Section 8.4 where we explore lifting operators from *computational social choice* to define extension-ranking semantics.

1.3 THESIS OUTLINE

The overall structure of this thesis is illustrate in Figure 1.4, which also shows the interrelationships between the chapters.

In Chapter 2, the foundational concepts and preliminaries are introduced. This chapter begins with a review of the general notion of a *relation* and then delves into the essential background for abstract argumentation, including abstract argumentation frameworks (AFs). It provides an overview of Dung's *extension semantics* (Dung,

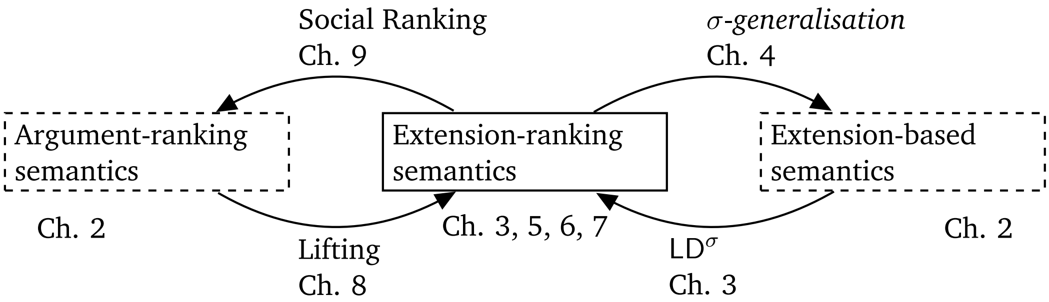

Figure 1.4: Thesis overview. Showing the connections between argument-ranking, extension-based, and extension-ranking semantics, where the dashed parts are not original content of this thesis. Each edge is discussed in a separate chapter.

1995) and introduces the argument-ranking semantics as proposed by Amgoud and Ben-Naim (2013).

Chapter 3 presents the first set of original contributions. It defines the notion of *extension-ranking semantics*, which maps an abstract argumentation framework to a preorder over all sets of arguments. This semantics enables the comparison of sets of arguments in terms of plausibility of acceptance. Additionally, the chapter demonstrates that any extension semantics can be expressed in terms of an extension-ranking semantics.

Chapter 4 introduces a set of principles for evaluating and developing extension-ranking semantics. Each principle addresses a specific aspect that a well-behaved extension-ranking semantics should satisfy.

In Chapter 5 argumentative reasoning is broken down into conceptually simple *base relations*. Each base relation focuses on a single aspect of argumentation, enabling the comparison of two sets of arguments according to that specific relation.

Chapter 6 combines the base relations to define families of extension-ranking semantics. Unlike argument-ranking semantics, these families properly generalise Dung's extension semantics, ensuring that the most plausible sets are acceptable under Dung's framework.

Chapter 7 explores an alternative approach to defining extension-ranking semantics by leveraging conditional logic. The chapter begins by reviewing the necessary background on conditional logic then introduce ordinal conditional functions for argumentation frameworks. Finally, it defines an extension-ranking semantics inspired by *System Z* and analyses its behaviour.

Next, this thesis explores the connection between argument-ranking semantics and extension-ranking semantics. In Chapter 8, various approaches to transforming or *lifting* argument-ranking semantics into extension-ranking semantics are discussed. The chapter begins by examining methods found in the argumentation literature and then explores approaches from the field of computational social choice.

The opposite direction of Chapter 8 is discussed in Chapter 9, transforming an extension-ranking semantics into an argument-ranking semantics. This transformation utilises *social ranking functions* from computational social choice. The chapter adopts a general perspective, presenting sufficient and necessary conditions for both the extension-ranking semantics and the social ranking function to satisfy the principles of argument-ranking semantics.

Chapter 10 investigate how computationally hard it is to rank sets of arguments. It introduces the problem of τ EXTENSION RANKING, which decides between two sets of arguments which set is better ranked according to an extension-ranking semantics τ. For many extension-ranking semantics, this problem is shown to be solvable by a deterministic polynomial-time algorithm, while for others, only parameterised results are presented.

Finally, Chapter 11 summarises the results of this thesis, discusses related work not covered in previous chapters, and outlines future research directions.

The main chapters do not include technical proofs. Instead, each chapter has a corresponding appendix section (Appendix A.2 to Appendix A.8) containing the technical proofs. Appendix B provides additional information about Example 148.

PRELIMINARIES 2

In this chapter, we introduce the necessary background information required for this thesis. We start by reviewing *relations* in general (Section 2.1), and then present the basic formalism of an *abstract argumentation framework* introduced by Dung (1995) (Section 2.2) and the related reasoning approaches (Section 2.3). We conclude by recalling *argument-ranking semantics* (Amgoud and Ben-Naim, 2013) (Section 2.4).

2.1 RELATIONS

A *relation* between sets describes the interaction or correspondence between these sets. In this thesis, we exclusively focus on relations that involve two sets, known as *binary relations*.

Definition 1. *Let X and Y be sets. We call $\mathcal{R} \subseteq X \times Y$ a* binary relation*.*

A well-known example of a binary relation is the "is greater than" ($>$) notion among natural numbers in mathematics. Another commonly used binary relation is the "is subset of" ($\subseteq$) relation between sets. Both the relation among natural numbers and the set relation are a special instance of a class of relations known as *order relations*. This class of relations imposes additional requirements. For instance, an order relation must be *transitive*. For natural numbers, this means that if $2 > 1$ and $3 > 2$, then it must also hold that $3 > 1$. Formally, these properties are defined as follows:

Definition 2. *Let X be a set and $\mathcal{R}$ a binary relation such that $\mathcal{R} \subseteq X \times X$. $\mathcal{R}$ is*

- reflexive *if $(x, x) \in \mathcal{R}$ for every $x \in X$;*
- transitive *if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$ for all $x, y, z \in X$;*
- total *if $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$ for all $x, y \in \mathcal{R}$;*
- antisymmetric *if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$ for $x, y \in X$.*

With these properties, we can identify several distinct types of binary relations.

Definition 3. *Let $\mathcal{R}$ be a binary relation. $\mathcal{R}$ is called:*

- preorder *if $\mathcal{R}$ is reflexive and transitive;*
- partial order *if $\mathcal{R}$ is reflexive, transitive, and antisymmetric;*
- weak order *if $\mathcal{R}$ is reflexive, transitive, and total;*
- linear order *if $\mathcal{R}$ is reflexive, transitive, total, and antisymmetric;*

For order relations such as $\geq$, we employ *infix notation*. That is, for the order relation $\mathcal{R}$ and elements x and y we write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$.

An extension of the order relation is the *lexicographic preference order*, which allows the comparison of two vectors of elements using a preorder over the individual elements.

Definition 4. *Let X be a set and $\mathcal{R}$ a preorder. Let $V = \langle V_1, V_2, \dots \rangle$ and $V' = \langle V'_1, V'_2, \dots \rangle$ be two (possibly infinite) real-valued vectors such that $V_1, V_2, \dots,$ $V'_1, V'_2, \dots \in X$.*

The lexicographic preference order $\succeq^{lex}_{\mathcal{R}}$ *is defined as:*

- *$V \succ^{lex}_{\mathcal{R}} V'$ if and only if there exists an i such that $(V_i, V'_i) \in \mathcal{R}$ and $(V'_i, V_i) \notin \mathcal{R}$, and for all for all $j < i$, $(V_j, V'_j) \in \mathcal{R}$ and $(V'_j, V_j) \in \mathcal{R}$.*
- *$V \simeq^{lex}_{\mathcal{R}} V'$ if and only if for all i, $(V_i, V'_i) \in \mathcal{R}$ and $(V'_i, V_i) \in \mathcal{R}$.*

The lexicographic preference order compares two vectors or any other ordered structure by comparing each pair of corresponding elements sequentially.

Example 4. *Consider the set of natural numbers $\mathbb{N}$ and the binary relation "is less than" $<$. Assume we have two vectors $V = \langle 1, 2, 3, 4 \rangle$ and $V' = \langle 1, 3, 5, 2 \rangle$. We begin by comparing the first elements of V and V', which are equal. We then proceed to the second position, where we find $2 < 3$. We conclude that $V \succ^{lex}_{<} V'$ and do not compare the subsequent positions of the vectors.*

We denote with $\mathcal{P}(X)$ the *powerset* of a set X. This is the set of all subsets of X.

Definition 5. *Let X be a set, the* powerset $\mathcal{P}(X)$ *is defined as:*

$$\mathcal{P}(X) = \{S \mid S \subseteq X\}$$

The powerset of X includes the empty set $\emptyset$ and the set X itself. If X has n elements, then $\mathcal{P}(X)$ has 2^n elements.

2.2 ABSTRACT ARGUMENTATION FRAMEWORKS

Abstract argumentation framework (Dung, 1995) is a formalism for reasoning with conflicting pieces of information, such as evidence in a criminal case or arguments in a discussion about holiday plans, using arguments and attacks between them.

Definition 6. *An* abstract argumentation framework *(AF) is a directed graph $F = (A, R)$, where A is a finite set of* arguments *and R is an* attack relation $R \subseteq A \times A$.

For an abstract argumentation framework $F = (A, R)$, an argument $a \in A$ is said to *attack* an argument $b \in A$ if $(a, b) \in R$. A set $E \subseteq A$ is said to *defend* an argument $a \in A$ if every argument $b \in A$ that attacks a is, in turn, attacked by some argument $c \in E$. For $a \in A$, we define:

$$a_F^- = \{b \mid (b, a) \in R\} \quad \text{and} \quad a_F^+ = \{b \mid (a, b) \in R\}.$$

In other words, a_F^- is the set of attackers of a and a_F^+ is the set of arguments attacked by a. For a set of arguments $E \subseteq A$, we extend these definitions to E_F^+ and E_F^- via:

$$E_F^+ = \bigcup_{a \in E} a_F^+ \quad \text{and} \quad E_F^- = \bigcup_{a \in E} a_F^-$$

If the abstract argumentation framework is clear from the context, we omit the index. Additionally, with a slight abuse of notation, we denote an attack from an argument on a set of arguments as follows: For an argumentation framework $F = (A, R)$ with $a \in A$, and $E, E' \subseteq A$, we denote with $(a, E) \in R$ that there exists an argument $b \in E$ such that $(a, b) \in R$. Extending this notation to (E, E'), then there exist arguments $b \in E$ and $c \in E'$ such that $(b, c) \in R$.

In Chapter 1, we have already presented three different abstract argumentation frameworks. In Example 1, we discus a murder case where John is the prime suspect, and the corresponding AF is depicted in Figure 1.1. The argument s attacks both argument a and itself, while argument a attacks argument g. In Example 2, argument v attacks argument b, which in turn attacks argument a, thereby defending a from the attack of b.

For two AFs $F = (A, R)$ and $F' = (A', R')$, we define their union by

$$F \cup F' = (A, R) \cup (A, R') = (A \cup A', R \cup R')$$

An *isomorphism* γ between two AFs $F = (A, R)$ and $F' = (A', R')$ is a bijective function $\gamma : A \to A'$ such that $(a, b) \in R$ if and only if $(\gamma(a), \gamma(b)) \in R'$ for all $a, b \in A$.

2.3 EXTENSION SEMANTICS

A number of semantic notions for reasoning with abstract argumentation frameworks can be found in the literature (for overviews see Baroni et al. (2018); Gabbay et al. (2021)). An common approach are *extension semantics*, where sets of arguments are considered jointly acceptable. That is, a set is acceptable, rather than individual arguments. To define *extension semantics*, we rely on the two fundamental concepts of *conflict-freeness* and *admissibility*.

Definition 7. *Given AF $F = (A, R)$, a set $E \subseteq A$ is*

- *a* conflict-free *set if and only if for all pairs $a, b \in E$, we have $(a, b) \notin R$.*
- *an* admissible *set if and only if E is conflict-free and E defends all its elements, i. e., $E^-_F \subseteq E^+_F$.*

We denote the sets of conflict-free and admissible sets of an abstract argumentation framework $F = (A, R)$ by $\mathsf{cf}(F)$ and $\mathsf{ad}(F)$, respectively. An important property of admissibility is the *Fundamental Lemma* by Dung (1995).

Lemma 1 (Dung (1995))**.** *Let $F = (A, R)$ be an AF and $E \in \mathsf{ad}(F)$. If $a \in A$ is defended by E, then $E \cup \{a\} \in \mathsf{ad}(F)$.*

Intuitively, an argument defended by an admissible set can be added to that admissible set without violating the admissibility of the set.

We define the *characteristic function* for abstract argumentation frameworks.

Definition 8. *For AF $F = (A, R)$, the* characteristic function $\mathfrak{F}_F : \mathcal{P}(A) \to \mathcal{P}(A)$ *is defined for $E \subseteq A$ via:*

$$\mathfrak{F}_F(E) = \{a \in A \mid E \text{ defends } a\}$$

The characteristic function, for a set of arguments E, identifies the arguments defended by E. We introduce the *extension semantics* proposed by Dung (1995).

Definition 9. *Let $F = (A, R)$ be an AF. An admissible set $E \subseteq A$ is*

- *a* complete *extension* (co) *if and only if $E = \mathfrak{F}_F(E)$;*
- *a* grounded *extension* (gr) *if and only if it is the $\subseteq$-minimal complete extension;*
- *a* preferred *extension* (pr) *if and only if it is a $\subseteq$-maximal admissible set;*
- *a* stable *extension* (st) *if and only if $E^+_F = A \setminus E$.*

Note that the grounded extension is unique for each abstract argumentation framework (Baroni et al., 2018). Furthermore, for each abstract argumentation framework, there is always at least one extension for the complete, preferred and grounded extension semantics. However, this is not necessarily the case for the stable extension semantics.

Example 5. *Consider the AF F_1 from Example 1 (recalled in Figure 2.1). In this AF, none of the arguments is part of a stable extension. This is because each argument either cannot be defended against its attackers (arguments a and g) or attacks itself (argument s). Consequently, F_1 has no conflict-free set which attacks every argument not contained in the set, and therefore, there is no stable extension.*

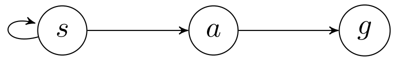

Figure 2.1: Recalled AF F_1 from Example 1.

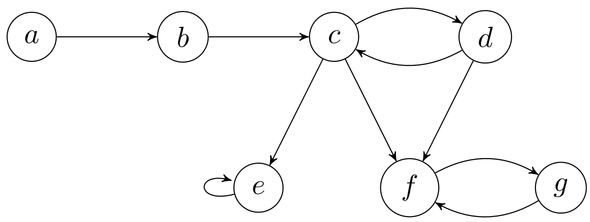

Figure 2.2: AF F_4 from Example 6.

Due to of the nonexistence of stable extensions, Caminada et al. (2012) proposed a novel variation of the stable extension semantics, the so-called *semi-stable* extension semantics.

Definition 10. *For AF $F = (A, R)$, a set $E \subseteq A$ is a* semi-stable *extension if and only if E is a complete extension where $E \cup E_F^+$ is maximal with respect to set inclusion.*

The set of semi-stable extensions coincides with the set of stable extensions when the latter is not empty (Caminada et al., 2012).

Example 6. *Consider the abstract argumentation framework F_4 depicted as a directed graph in Figure 2.2. F_4 has four complete extensions:*

$$E_1 = \{a\}, \qquad E_3 = \{a, c, g\},$$
$$E_2 = \{a, g\}, \quad E_4 = \{a, d, g\}.$$

$\{a\}$ is the grounded extension, while $\{a, c, g\}$ and $\{a, d, g\}$ are both preferred extensions, but only $\{a, c, g\}$ is a stable extension and therefore also a semi-stable extension.

The sets of extensions of an abstract argumentation framework F, for the five semantics mentioned, are represented by $\mathsf{co}(F)$, $\mathsf{gr}(F)$, $\mathsf{pr}(F)$, $\mathsf{st}(F)$, and $\mathsf{sst}(F)$. These extension semantics are commonly referred to as *Dung's extension semantics* for the set of extensions with σ-extension for $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$. A more detailed discussion of these semantics can be found in (Dung, 1995; Baroni et al., 2018).

Using the extension semantics, we can determine whether each argument is accepted or not. If an argument a is part of at least one σ-extension of an abstract argumentation framework F, then a is *credulously accepted* with respect to the extension semantics σ. If a is part of every σ-extension of F, then a is *skeptically accepted* with respect to the extension semantics σ. An argument a that is not part of any σ-extension is *rejected* with respect to the extension semantics σ in F. We denote the set of credulously accepted, skeptically accepted and rejected arguments in AF F with respect to the extension semantics σ as $\mathsf{cred}_\sigma(F)$, $\mathsf{skep}_\sigma(F)$ and $\mathsf{rej}_\sigma(F)$, respectively.

2.4 ARGUMENT-RANKING SEMANTICS

While extension semantics allow us to classify arguments as skeptically accepted, credulously accepted, or rejected, this classification is not expressive enough.

Example 7. *Let us discuss Example 1 further. For the judge to determine John's guilt, they need to identify an admissible set that reflects their judgment. However, the only admissible set available is the empty set. Therefore, the judge requires more options to evaluate and compare the presented arguments, determining which is "stronger" and should be considered for the final decision.*

Argument-ranking semantics (also known as *ranking-based semantics*) were introduced by Amgoud and Ben-Naim (2013) and aim to rank individual arguments within an abstract argumentation framework. With these semantics we can determine that one argument is "better" or "stronger" than another.

Definition 11. *An* argument-ranking semantics ρ *is a function, which maps an AF* $F = (A, R)$ *to a preorder* $\succeq_F^\rho$ *on* A*.*

Intuitively, $a \succeq_F^\rho b$ means, that a is at least as *strong* as b in F. We define the usual abbreviations as follows:

- $a \succ_F^\rho b$ denotes *strictly stronger*, meaning $a \succeq_F^\rho b$ and $b \not\succeq_F^\rho a$;
- $a \simeq_F^\rho b$ denotes *equally strong*, meaning $a \succeq_F^\rho b$ and $b \succeq_F^\rho a$;
- $a \bowtie_F^\rho b$ denotes *incomparability*, meaning neither $a \succeq_F^\rho b$ nor $b \succeq_F^\rho a$.

Example 8. *Continuing with Example 7, the argument s attacks itself, making it unreliable. Therefore, the judge should disregard the witness's testimony, as their argument is weak. Consequently, argument a is only attacked by this weak argument s, making a stronger than both s and g, since a also attacks g.*

By ranking the arguments, we find that a is the strongest argument, followed by g, with s being the weakest, i. e.:

$$a \succ g \succ s$$

This ranking leads the judge to be more inclined to rule "not guilty", as the reasons against John's guilt are stronger than the arguments supporting it.

Example 8 illustrates the use of argument rankings. While extension semantics alone did not allow the judge to decide on John's guilt, ranking the arguments enabled different judgment.

A subfamily of argument-ranking semantics are *gradual semantics* (Cayrol and Lagasquie-Schiex, 2005), where each argument is assigned a numerical strength value. In this paper we consider every gradual semantics to be *positive* and *increasing*, this means that every strength value is non-negative and the higher the strength value is the better an argument is ranked. For an AF $F = (A, R)$ and a gradual semantics ρ, we denote the numerical strength value of an argument $a \in A$ by $\rho_F(a)$. Note that we normalise the values to $[0, 1]$, so $\rho_F(a) \to [0, 1]$ for $a \in A$.

2.4.1 Various argument-ranking semantics

Several argument-ranking semantics have been proposed in the literature (for a comprehensive overview, see (Bonzon et al., 2016)). In this work, we consider three specific semantics, *burden-based argument-ranking semantics* (Amgoud and Ben-Naim, 2013), *h-categoriser argument-ranking semantics* (Besnard and Hunter, 2001), and *serialisability argument-ranking semantics* (Blümel and Thimm, 2022).

Burden-based Argument-ranking Semantics

The *burden-based argument-ranking semantics* proposed by Amgoud and Ben-Naim (2013) assesses the strength of an argument relative to the strength of its attackers. So, the stronger an attacker is, the weaker the argument becomes.

Definition 12. *Let $F = (A, R)$ be an AF. The* burden number $\mathsf{bur}_i(a)$ *for argument $a \in A$ in iteration $i \in \mathbb{N}$ is defined as follows:*

$$\mathsf{bur}_i(a) := \begin{cases} 1 & \text{if } i = 0 \\ 1 + \sum_{b \in a_F^-} \frac{1}{\mathsf{bur}_{i-1}(b)} & \text{otherwise} \end{cases}$$

Let $\mathsf{bur}(a) = (\mathsf{bur}_0(a), \mathsf{bur}_1(a), \mathsf{bur}_2(a), \dots)$ *be the* burden vector *for $a \in A$, and let* $\mathsf{bur}(b)$ *be the corresponding burden vector for $b \in A$.*

We define the burden-based argument-ranking semantics *(Bbs) with the lexicographic preference order with respect to the binary relation of "is less than" $<$:*

$$a \succeq_F^{Bbs} b \textit{ if and only if } \mathsf{bur}(a) \succeq_<^{lex} \mathsf{bur}(b)$$

For two arguments a and b, the *burden-based argument-ranking semantics* compares the strength of their attackers. In case of equivalence, it considers the attackers of the attackers.

Example 9. *Consider AF F_4 from Example 6. Since argument a is unattacked, we have* $\mathsf{bur}(a) = (1, 1, \dots)$. *For argument g, which has one attacker f, we compute:*

- *For $i = 1$, $\mathsf{bur}_1(g) = 2$, since $\mathsf{bur}_0(f) = 1$.*
- *For the next iteration, we need to compute $\mathsf{bur}_1(f)$. Since f has three attackers, we have:*
$$\mathsf{bur}_1(f) = 1 + \frac{1}{\mathsf{bur}_0(c)} + \frac{1}{\mathsf{bur}_0(d)} + \frac{1}{\mathsf{bur}_0(g)} = 4$$
Thus,
$$\mathsf{bur}_2(g) = \frac{5}{4}$$

For argument d, we have $\mathsf{bur}_1(d) = 2$. To differentiate d and g, we need to go one step further to $i = 2$

$$\mathsf{bur}_2(d) = 1 + \frac{1}{\mathsf{bur}_1(c)} = 1 + \frac{1}{3} = \frac{4}{3}$$

Therefore, we have $g \succ^{Bbs}_{F_4} d$*, since* $\frac{5}{4} < \frac{4}{3}$*. The corresponding argument ranking is:*

$$a \succ^{Bbs}_{F_4} g \succ^{Bbs}_{F_4} d \succ^{Bbs}_{F_4} b \succ^{Bbs}_{F_4} e \succ^{Bbs}_{F_4} c \succ^{Bbs}_{F_4} f$$

h-Categoriser Argument-ranking Semantics

One example of an gradual semantics is the *h-categoriser argument-ranking semantics* proposed by Besnard and Hunter (2001). Similar to Bbs, the h-categoriser argument-ranking semantics considers the strength of the attackers to calculate the strength of an argument.

Definition 13. *Let* $F = (A, R)$ *be an AF and* $a \in A$*. A* h-categoriser function $\mathsf{cat} : A \to (0, 1]$ *is a function that satisfies:*

$$\mathsf{cat}(a) = \frac{1}{1 + \sum_{b \in a^-_F} \mathsf{cat}(b)}$$

We define the h-categoriser argument-ranking semantics *(*Cat*) as:*

$$a \succeq^{Cat}_{F} b \textit{ if and only if } \mathsf{cat}(a) \geq \mathsf{cat}(b)$$

Pu et al. (2014) have shown that the h-categoriser argument-ranking semantics is well-defined, this means there is an h-categoriser function that is unique for each abstract argumentation framework.

Example 10. *Consider AF* F_4 *from Example 6. As argument* a *is unattacked,* $\mathsf{cat}(a) = 1$*. For argument* b*, we get* $\mathsf{cat}(b) = \frac{1}{1+\mathsf{cat}(a)} = 0.5$*, since* a *is the only attacker of* b*. To calculate* $\mathsf{cat}(c)$ *and* $\mathsf{cat}(d)$*, we solve the system of equations:*

$$\mathsf{cat}(c) = \frac{1}{1 + \mathsf{cat}(b) + \mathsf{cat}(d)} \qquad \mathsf{cat}(d) = \frac{1}{1 + \mathsf{cat}(c)}$$

Solving these, we find $\mathsf{cat}(c) = 0.46$ *and* $\mathsf{cat}(d) = 0.69$*. Similar calculations are performed for the remaining arguments. The resulting argument ranking with respect to the h-categoriser argument-ranking semantics is:*

$$a \succ^{Cat}_{F_4} g \succ^{Cat}_{F_4} d \succ^{Cat}_{F_4} e \succ^{Cat}_{F_4} b \succ^{Cat}_{F_4} c \succ^{Cat}_{F_4} f$$

Serialisability Argument-ranking Semantics

Next, we revisit *serialisability argument-ranking semantics* proposed by Blümel and Thimm (2022), which is conceptually distinct from the burden-based argument-ranking semantics and h-categoriser argument-ranking semantics. While Bbs and Cat focus on the strength of the attackers, the serialisability argument-ranking semantics ranks arguments according to the minimal number of conflicts that iteratively must to be resolved before these arguments can be included in an admissible set. To define this semantics, we need to introduce a few concepts.

First, we identify the *initial sets* of an abstract argumentation framework (Xu and Cayrol, 2016; Thimm, 2022). These sets are the subset minimally admissible sets.

Definition 14. *Let $F = (A, R)$ be an AF and $E \subseteq A$ a set of arguments with $E \neq \emptyset$. E is an* initial set *if and only if $E \in \mathsf{ad}(F)$ and there is no $E' \subset E$ with $E' \neq \emptyset$ such that $E' \in \mathsf{ad}(F)$.*

We denote by $\mathsf{IS}(F)$ the set of initial sets of abstract argumentation framework F. Next, we define the notion of an *E-reduct* (Baumann et al., 2020). These are the abstract argumentation frameworks from which all arguments of the set E and the arguments attacked by E are removed.

Definition 15. *Let $F = (A, R)$ be an AF and $E \subseteq A$. We define the E-*reduct *F^E via:*

$$F^E = (A \setminus (E \cup E_F^+), R \cap (A' \times A')) \text{ with } A' = A \setminus (E \cup E_F^+)$$

Using the definitions of initial sets and E-reduct, we define a *serialisation sequence* for an abstract argumentation framework F.

Definition 16. *Let $F = (A, R)$ be an AF. We define the* serialisation sequence *for F as a sequence $\mathcal{S} = (E_1, \ldots, E_n)$ with $E_1 \in \mathsf{IS}(F)$ and for each $2 \leq i \leq n$ it holds that $E_i \in \mathsf{IS}(F^{E_1 \cup \cdots \cup E_{i-1}})$.*

A *serialisation sequence* is a sequence of sets of arguments which are admissible in the reduct with respect to all sets appearing previously in the sequence. Thimm (2022) has shown that a serialisation sequence $(E_1, \ldots, E_n)$ induces an admissible set $E = E_1 \cup \cdots \cup E_n$ and for every admissible set, there is at least one such sequence. We can identify for each argument the shortest sequence such that this argument is contained in an admissible set.

Definition 17. *Let $F = (A, R)$ be an AF and $a \in A$. We define the* serialisation index *$\mathsf{seri}_F(a)$ for argument a in AF F via:*

$$\mathsf{seri}_F(a) = \mid \min\{n \mid (E_1, \ldots, E_n) \text{ is a serialisation sequence of } F \text{ and } a \in E_n\} \mid$$

with $\min \emptyset = \infty$.

The serialisation index for argument a is the minimum number of conflicts that need to be eliminated in order for a to be credulously accepted with respect to admissibility, and thus represents a value by which arguments can be ranked.

Definition 18. *Let $F = (A, R)$ be an AF and $a, b \in A$. We define the* serialisability argument-ranking semantics *(Ser) via:*

$$a \succeq_F^{Ser} b \text{ if and only if } \mathsf{seri}_F(a) \leq \mathsf{seri}_F(b)$$

Example 11. *Consider again abstract argumentation framework F_4 from Example 6. The sets $\{a\}, \{d\}$ and $\{g\}$ are the initial sets of F_4 and thus $\mathsf{seri}_{F_4}(a) = \mathsf{seri}_{F_4}(d) = \mathsf{seri}_{F_4}(g) = 0$. For argument c, we remove $\{a\}$ and therefore argument b, making $\{c\}$ admissible in the reduct $F_4^{\{a\}}$. The arguments b, e and f are reject with respect to admissibility, so we have $\mathsf{seri}_{F_4}(b) = \mathsf{seri}_{F_4}(e) = \mathsf{seri}_{F_4}(f) = \infty$. The full argument ranking is:*

$$g \simeq_{F_4}^{Ser} d \simeq_{F_4}^{Ser} a \succ_{F_4}^{Ser} c \succ_{F_4}^{Ser} f \simeq_{F_4}^{Ser} e \simeq_{F_4}^{Ser} b$$

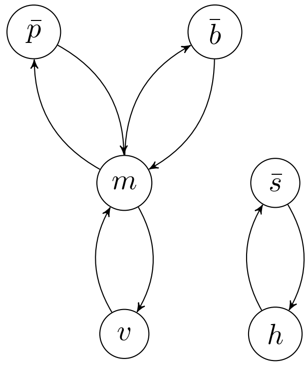

Figure 2.3: Recalled AF F_3 for Example 3.

2.4.2 Principles

To guide the development of new argument-ranking semantics, several principles have been proposed. Each principle represents a distinct property for argument rankings. For more detailed information on these principles, we refer the reader to Bonzon et al. (2016) and Amgoud and Ben-Naim (2013). Before exploring these principles, we need to introduce some additional notions and concepts. We start with some notions known from *graph theory*.

Definition 19. *Let $F = (A, R)$ be an AF with arguments $a, b \in A$. A* directed path *$P(a, b)$ of length $l_P = n$ between two arguments a, b is a sequence of arguments $P(a, b) = (a_0, a_1, ..., a_n)$ where $(a_i, a_{i+1}) \in R$ for all i, with $a_0 = a$ and $a_n = b$.*

A undirected path *$P_u(a, b)$ of length $l_{P_u} = n$ between two arguments a, b is a sequence of arguments $P_u(a, b) = (a_0, a_1, ..., a_n)$ such that for every i, either $(a_i, a_{i+1}) \in R'$ or $(a_{i+1}, a_i) \in R'$.*

Example 12. *Consider AF F_4 from Example 6. From argument a, we can reach each argument with at least one directed path. For instance, to reach argument d from a, we have the sequence: (a, b, c, d). However, the reverse is not possible; we cannot reach argument a from d.*

The notion of *connected components* is closely related to the notion of a path. A connected component is a subgraph where each pair of arguments has an undirected path between them.

Definition 20. *The* connected components *$cc(F)$ of an AF F are the maximal subgraphs $F' = (A', R')$ where, for every pair of arguments $a, b \in A'$, there exists an* undirected path *$P_u(a, b) = (a = a_0, a_1, ..., a_{n-1}, a_n = b)$.*

Example 13. *Consider AF F_3 from Example 3 recalled in Figure 2.3. F_3 has two connected components: the first one contains arguments $m, v, \bar{p}$, and $\bar{b}$, and the second one consist out of h and $\bar{s}$.*

In contrast, AF F_4 from Example 6 has only one connected component, as argument a has a path to every other argument, ensuring that every pair of arguments has an undirected path between them.

The final set of notions pertains to defence.

Definition 21. *Let $F = (A, R)$ be an AF. For an extension semantics σ, an argument a* weakly σ-supports *b if $b \in \mathsf{cred}_\sigma(F)$ and for all $E \in \sigma(F)$, with $b \in E$, it follows that $a \in E$. An argument a* strongly σ-supports *b if $b \in \mathsf{cred}_\sigma(F)$ and for all $E \in \sigma(F)$ with $b \in E$, there exists $E' \in \sigma(F)$ such that $E' \subseteq E$, $a \in E'$ and $b \notin E'$.*

Informally, if argument a *weakly* σ-supports argument b, then every σ-extension containing b also contains a. If a *strongly* σ-supports argument b, then for every σ-extension containing both b and a, there must be a smaller σ-extension containing a and not b. Blümel and Thimm (2022) have shown that strong σ-support implies weak σ-support.

Example 14. *Consider F_4 from Example 6 and its corresponding complete extensions: $\{a\}$, $\{a, g\}$, $\{a, c, g\}$, and $\{a, d, g\}$. Arguments c and d are strongly* co*-supported by a and g. Any* co*-extension containing c or d ($\{a, c, g\}$ and $\{a, d, g\}$) also contains a and g. Additionally, there is a smaller* co*-extension $\{a, g\}$ that does not include c or d.*

Now, we introduce principles for argument-ranking semantics.

Definition 22. *An argument-ranking semantics ρ satisfies the respective principle if and only if for all AFs $F = (A, R)$ and any $a, b \in A$:*

Abstraction (*Abs*) (Amgoud and Ben-Naim, 2013). *The names of the arguments do not affect the ranking.*

For a pair of AFs $F = (A, R)$ and $F' = (A', R')$ and every isomorphism $\gamma : A \to A'$, we have $a \succeq^\rho_F b$ if and only if $\gamma(a) \succeq^\rho_{F'} \gamma(b)$.

Independence (*In*) (Matt and Toni, 2008; Amgoud and Ben-Naim, 2013). *Unconnected arguments should not influence a ranking.*

For every $F' = (A', R') \in cc(F)$ and for all $a, b \in A'$: $a \succeq^\rho_F b$ if and only if $a \succeq^\rho_{F'} b$.

Void Precedence (*VP*) (Matt and Toni, 2008; Amgoud and Ben-Naim, 2013). *Unattacked arguments should be ranked better than attacked ones.*

If $a^-_F = \emptyset$ and $b^-_F \neq \emptyset$ then $a \succ^\rho_F b$.

Self-Contradiction (*SC*) (Matt and Toni, 2008). *Self-attacking arguments should be ranked worse than any other argument.*

If $(a, a) \notin R$ and $(b, b) \in R$ then $a \succ^\rho_F b$.

Cardinality Precedence (*CP*) (Amgoud and Ben-Naim, 2013). *An argument with less attackers is ranked better than one with more attackers.*

If $|a_F^-| < |b_F^-|$ *then* $a \succ_F^\rho b$.

Quality Precedence (*QP*) (Amgoud and Ben-Naim, 2013). *If there is an attacker of an argument which is ranked better than every attacker of an other argument, the latter is ranked better than the former.*

If there is $c \in b_F^-$ *such that for all* $d \in a_F^-$ *it holds that* $c \succ_F^\rho d$ *then* $a \succ_F^\rho b$.

Counter-Transitivity (*CT*) (Amgoud and Ben-Naim, 2013). *If the attackers of an argument are as numerous and as strong as the attackers of another argument, then the former is as least as strong as the latter.*

If some injecitve $f : a_F^- \to b_F^-$ *exists such that* $f(x) \succeq_F^\rho x$ *for all* $x \in a_F^-$, *then* $a \succeq_F^\rho b$.

Strict Counter-Transitivity (*SCT*) (Amgoud and Ben-Naim, 2013). *Strict version of* CT.

If some injecitve $f : a_F^- \to b_F^-$ *exists such that* $f(x) \succeq_F^\rho x$ *for all* $x \in a_F^-$ *and either* $|a_F^-| < |b_F^-|$ *or there exists some* $x \in a_F^-$ *with* $f(x) \succ_F^\rho x$, *then* $a \succ_F^\rho b$.

Defense Precedence (*DP*) (Amgoud and Ben-Naim, 2013). *For two arguments with the same number of attackers, a defended argument is ranked better than a non-defended argument.*

If $|a_F^-| = |b_F^-|$, $(a_F^-)_F^- \neq \emptyset$ *and* $(b_F^-)_F^- = \emptyset$, *then* $a \succ_F^\rho b$.

Distributed Defense Precedence (*DDP*) (Amgoud and Ben-Naim, 2013). *If every attacker of argument is counterattacked by exactly one defender, then this argument is ranked better than an argument for which there is an attacker attacked by more than one defender.*

If $|a_F^-| = |b_F^-|$ *and* $|(a_F^-)_F^-| = |(b_F^-)_F^-|$, *and if the defence of* a *is simple – every direct defender of* a *directly attacks exactly one direct attacker of* a *– and distributed – every direct attacker of* a *is attacked by at most one argument – and the defence of* b *is simple but not distributed, then* $a \succ_F^\rho b$.

Non-attacked Equivalence (*NaE*) (Bonzon et al., 2016) *Two unattacked arguments should be equally ranked.*

If $a_F^- = b_F^- = \emptyset$, *then* $a \simeq_F^\rho b$.

Attack vs. Full Defense (*AvsFD*) (Bonzon et al., 2016). *Arguments with no unattacked indirect attackers should be ranked better than arguments attacked only by one unattacked argument.*

If F *is acyclic and every path* $P(u, a)$ *in* F *from unattacked* u *to* a *has* $l_p = 0$ *mod* 2 *and there exists unattacked* $v \in b_F^-$, *then* $a \succ_F^\rho b$.

σ-Compatibility (*σ-C*) (Blümel and Thimm, 2022). *Credulously accepted arguments should be ranked better than rejected arguments.*

For an extension semantics σ it holds that if $a \in \mathsf{cred}_\sigma(F)$ and $b \in \mathsf{rej}_\sigma(F)$, then $a \succ^\rho_F b$.

σ-sk-Compatibility (*σ-sk-C*) (Bengel et al., 2024a). *Skeptically accepted arguments should be ranked better than non-skeptically accepted arguments.*

For an extension semantics σ it holds that if $a \in \mathsf{skep}_\sigma(F)$ and $b \notin \mathsf{skep}_\sigma(F)$, then $a \succ^\rho_F b$.

weak σ-Support (*wσ-S*) (Blümel and Thimm, 2022). *If an argument a is an unavoidable side effect of accepting another argument b, then a should be at least as acceptable as b.*

If a weakly σ-supports b, then $a \succeq^\rho_F b$.

strong σ-Support (*sσ-S*) (Blümel and Thimm, 2022). *If an argument a is a prerequisite for accepting another argument b, and b is irrelevant for accepting a, then a should be ranked better than b.*

If a strongly σ-supports b, then $a \succ^\rho_F b$.

Universally Realisable (*UR*) (Skiba et al., 2022) *For every possible argument ranking there is an AF such that this ranking is induced by ρ.*

For any argument ranking r there exists an AF F' such that $\rho(F') = r$.

Note that these principles are not always compatible with each other, this means that there are principles that no argument-ranking semantics can satisfy simultaneously. Specially *SC* and *CP* are not compatible (Amgoud and Ben-Naim, 2013). Most results regarding the satisfied and violated principles of Bbs, Cat, and Ser can be found in the literature (Bonzon et al., 2016; Blümel and Thimm, 2022; Skiba et al., 2022; Bengel et al., 2024a). However, it remains to be shown that Ser violates *UR*.

Example 15. *Consider the preorder $r := a \succ b \succ c \succ d$. Suppose there is an AF F such that $Ser(F) = r$. This implies:*

$$\mathsf{seri}_F(a) < \mathsf{seri}_F(b) < \mathsf{seri}_F(c) < \mathsf{seri}_F(d)$$

This condition suggests that $\{a\}$ is the only initial set of F. Argument b must be attacked by at least one argument, and a must attack that argument, making $\{b\} \in \mathsf{IS}(F^{\{a\}})$. For $\mathsf{seri}_F(b) < \mathsf{seri}_F(c)$ to hold, c must be attacked by more arguments than b. Argument a cannot attack c, as this would prevent c from being defended. The only remaining argument is b. However, if b attacks c, then c is no longer part of $F^{\{a,b\}}$. This implies that c cannot be attacked by at least two arguments, thus r is not Ser-realisable.

In summary, we introduced the necessary background information on relations, abstract argumentation, extension semantics and argument-ranking semantics. The principles satisfied and violated by Bbs, Cat, and Ser are reported in Table 2.1.

	Bbs	Cat	Ser
Abs	✓	✓	✓
In	✓	✓	✓
VP	✓	✓	✗
SC	✗	✗	✗
CP	✓	✗	✗
QP	✓	✗	✗
CT	✓	✓	✗
SCT	✓	✓	✗
DP	✓	✓	✗
DDP	✓	✗	✗
NaE	✓	✓	✓
AvsFD	✗	✗	✓
σ-*C*	✗	✗	✓($\sigma = \mathsf{ad}$)
σ-*sk*-*C*	✗	✗	✗
wσ-*S*	✗	✗	✓($\sigma = \mathsf{ad}$)
sσ-*S*	✗	✗	✓($\sigma = \mathsf{ad}$)
UR	✓	✓	✗

Table 2.1: Principles satisfied and violated by Bbs, Cat, and Ser. Results taken from Bonzon et al. (2016), Blümel and Thimm (2022), Skiba et al. (2022) and Bengel et al. (2024a).

EXTENSION-RANKING SEMANTICS 3

In this chapter, we introduce a new kind of semantics that offers more expressiveness than binary classification. To achieve this, we consider preorders over sets of arguments. Since it is not always reasonable to enforce a decision between two sets of arguments, we do not necessarily enforce the totality of our relation, thus we only utilise preorders.

Definition 23. *Let $F = (A, R)$ be an AF. An* extension ranking *on F is a preorder $\sqsupseteq$ over the power set $\mathcal{P}(A)$ of all arguments. An* extension-ranking semantics τ *is a function that maps F to an extension ranking $\sqsupseteq^\tau_F$ on F.*

For an abstract argumentation framework $F = (A, R)$, an extension-ranking semantics τ, an extension ranking $\sqsupseteq^\tau_F$, consider sets $E, E' \subseteq A$. If $E \sqsupseteq^\tau_F E'$, we say that E is *at least as plausible to be accepted as* E' with respect to τ in F. We introduce the usual abbreviations:

- E is *strictly more plausible to be accepted than* E', denoted $E \sqsupset^\tau_F E'$, if $E \sqsupseteq^\tau_F E'$ but not $E' \sqsupseteq^\tau_F E$;
- E and E' are *equally plausible to be accepted*, denoted $E \equiv^\tau_F E'$, if $E \sqsupseteq^\tau_F E'$ and $E' \sqsupseteq^\tau_F E$;
- E and E' are *incomparable*, denoted $E \asymp^\tau_F E'$, if neither $E \sqsupseteq^\tau_F E'$ nor $E' \sqsupseteq^\tau_F E$.

Example 16. *Consider again F_4 from Example 6 (recalled in Figure 3.1). As previously shown $E_1 = \{a\}$, $E_2 = \{a, g\}$, $E_3 = \{a, c, g\}$, and $E_4 = \{a, d, g\}$ are the complete extensions. However, all these sets are treated equally in a reasoning process based on complete semantics. Only by employing a different extension semantics, such as preferred or stable extension semantics, can these sets be differentiated in terms of their plausibility of acceptance.*

When examining sets that are not accepted with respect to complete extension semantics, such as $\{d\}$ or $\{a, b\}$, these sets are considered equal under with respect to the complete extension semantics. However, $\{d\}$ is an admissible set, whereas $\{a, b\}$ is not

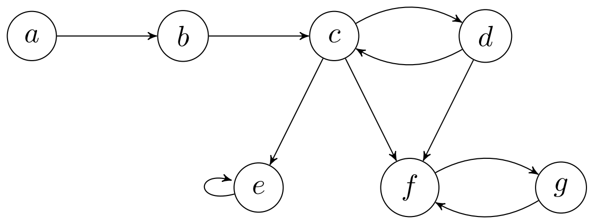

Figure 3.1: Recalled AF F_4 from Example 6.

even conflict-free. Therefore, it is reasonable to assert that $\{d\}$ is more plausible to be accepted than $\{a, b\}$. Thus, for an extension-ranking semantics ρ:

$$\{d\} \sqsupset^{\rho}_{F_4} \{a, b\}$$

The connection between extension-ranking semantics and extension semantics can be established. Extension semantics define a very naive instance of extension-ranking semantics as follows:

Definition 24. *Let $F = (A, R)$ be an AF. Given an extension semantics σ, we define the* least-discriminating extension-ranking semantics *with respect to σ, denoted LD^{σ}, by:*

$$E \sqsupset^{\mathsf{LD}^{\sigma}}_{F} E' \text{ if } E \in \sigma(F) \text{ and } E' \notin \sigma(F),$$
$$\text{and } E \equiv^{\mathsf{LD}^{\sigma}}_{F} E', \text{ if } E, E' \in \sigma(F) \text{ or } E, E' \notin \sigma(F).$$

Each σ-extension is ranked above every non-accepted set with respect to LD^{σ}. The least-discriminating extension-ranking semantics always gives a binary classification.

Example 17. *Consider AF F_4 from Example 6, with the complete extensions $E_1 = \{a\}$, $E_2 = \{a, g\}$, $E_3 = \{a, c, g\}$, and $E_4 = \{a, d, g\}$. These complete extensions are equally plausible to be accepted with respect to the least-discriminating extension-ranking semantics regarding* co*, i. e.:*

$$\{a\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, g\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, c, g\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, d, g\}$$

All other sets are not accepted with respect to the complete extension semantics and are therefore ranked below all co*-extensions, but these sets are ranked as equally plausible to accept. For example, the two rejected sets $\{d\}$ and $\{a, b\}$ are equally plausible to be accepted, i. e., $\{d\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, b\}$. Thus, we can extend the ranking from above:*

$$\{a\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, g\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, c, g\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, d, g\} \sqsupset^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{d\} \equiv^{\mathsf{LD}^{\mathsf{co}}}_{F_4} \{a, b\}$$

Note that the least-discriminating extension-ranking semantics is reflexive and transitive for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proposition 1. LD^{σ} *is reflexive and transitive for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.*

Proof: P. 167

To further explore the relationship between extension-ranking semantics and extension semantics, we identify the most plausible sets of an extension ranking, or the highest-ranked sets in the preorder.

Definition 25. *Let $F = (A, R)$ be an AF. We denote by $\mathtt{max}_\tau(F)$ the maximal (or* most plausible*) elements of the extension ranking $\sqsupseteq_F^\tau$, defined as:*

$$\mathtt{max}_\tau(F) = \{E \subseteq A \mid \nexists E' \subseteq A \textit{ with } E' \sqsupset_F^\tau E'\}$$

A set E is among the most plausible sets with respect to an extension-ranking semantics τ in an abstract argumentation framework F, if there is no strictly more plausible set.

Example 18. *Continuing from Example 6, consider the least-discriminating extension-ranking semantics* $\mathsf{LD}^{\mathsf{co}}$ *with respect to the complete semantics* co*, we get:*

$$\mathtt{max}_{\mathsf{LD}^{\mathsf{co}}}(F_4) = \{\{a\}, \{a, g\}, \{a, c, g\}, \{a, d, g\}\}$$

Every other set is ranked below these sets, making the complete extensions the most plausible sets. If we consider the least-discriminating extension-ranking semantics $\mathsf{LD}^{\mathsf{pr}}$ *with respect to the preferred semantics* pr*, we get:*

$$\mathtt{max}_{\mathsf{LD}^{\mathsf{pr}}}(F_4) = \{\{a, c, g\}, \{a, d, g\}\}$$

In this chapter, we introduced the notion of extension-ranking semantics, which ranks sets of arguments within an abstract argumentation framework. This ranking is more expressive than the binary classification of acceptable and non-acceptable sets induced by extension semantics. Additionally, we demonstrated the connection between extension-ranking semantics and extension semantics by defining an extension-ranking semantics based on an extension semantics.

PRINCIPLES 4

To assess and compare extension-ranking semantics, we introduce several general principles that describe various aspects of an intuitively well-behaved extension-ranking semantics. Some of these principles are adapted from existing literature on argument-ranking semantics, as discussed in Chapter 2, while others are specifically designed for extension-ranking semantics.

Classical extension semantics already possess many desirable properties for argumentative evaluation (van der Torre and Vesic, 2017). Therefore, the formalisation of extension-ranking semantics should align with these classical extension semantics, ensuring that the constraints of acceptable and non-acceptable sets of arguments are reflected in the extension ranking. We consider two aspects:

1. For a given extension semantics σ, we expect that only σ-extensions can be among the most plausible sets.

2. Since different σ-extensions are indistinguishable by σ, all σ-extensions should be among the most plausible sets.

We formalise the first requirement through a principle called *σ-soundness* and the second through *σ-completeness*. Combining these two principles gives us the principle of *σ-generalisation*.

Definition 26. *Let σ be an extension semantics and τ an extension-ranking semantics. τ satisfies*

- σ-soundness *if and only if for all AFs F:* $\mathtt{max}_\tau(F) \subseteq \sigma(F)$.

- σ-completeness *if and only if for all AFs F:* $\mathtt{max}_\tau(F) \supseteq \sigma(F)$.

- σ-generalisation *if and only if τ satisfies both* σ-soundness *and* σ-completeness.

For an extension-ranking semantics that satisfies *σ-generalisation*, the best-ranked sets, and thus the most plausible sets, coincide with the set of σ-extensions of the given abstract argumentation framework.

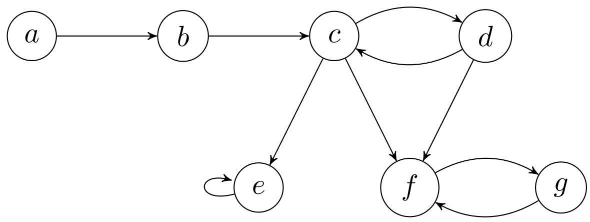

Figure 4.1: Recalled AF F_4 from Example 6.

Example 19. *Consider F_4 from Example 6 (recalled in Figure 4.1) and let τ be an extension-ranking semantics. If τ satisfies* co-*generalisation, then the most plausible sets are:*

$$\texttt{max}_\tau(F_4) = \{\{a\}, \{a, g\}, \{a, c, g\}, \{a, d, g\}\}$$

If τ satisfies pr-*generalisation, then the most plausible sets are:*

$$\texttt{max}_\tau(F_4) = \{\{a, c, g\}, \{a, d, g\}\}$$

Conflict-freeness is a desirable property in formal argumentation, as it ensures that no agent in a discussion contradicts themself. Therefore, we define the principle of *respect conflicts* which states that every conflict-free set should be strictly more plausible to accept than conflicting ones. This principle is similar to the *SC* principle of argument-ranking semantics, where self-contradicting arguments are considered the worst arguments.

Definition 27. *Let τ be an extension-ranking semantics. τ satisfies* respect conflicts *if and only if for all AFs $F = (A, R)$ with $E, E' \in A$ it holds that $E \in \mathsf{cf}(F)$ and $E \notin \mathsf{cf}(F)$ implies $E \sqsupset^\tau_F E'$.*

Example 20. *Consider AF F_4 from Example 6. The argument e is self-attacking, making the set $\{e\}$ strictly less plausible to accept than the grounded extension $\{a, c, g\}$ with respect to an extension-ranking semantics τ satisfying* respect conflicts*, thus:*

$$\{a, c, g\} \sqsupset^\tau_{F_4} \{e\}$$

Respect conflicts enforces a strict separation between conflict-free and conflicting sets, ensuring that cf-*soundness* is satisfied.

Proposition 2. *If extension-ranking semantics τ satisfies* respect conflicts*, then τ satisfies* cf-soundness. *Proof: P. 167*

The principles *respect conflicts* does not imply cf-*completeness* or cf-*soundness* for other combinations. This is because cf-*completeness* and cf-*soundness* only concern the most plausible sets. Additionally, *respect conflicts* does not impose any relationship between two conflict-free sets, unlike cf-*completeness*.

Example 21. *Consider again F_4 from Example 6. Let τ be the extension-ranking semantics that ranks all sets as equally plausible. While τ satisfies* cf-completeness*, it violates* respect conflicts *because the conflict-free set $\{a\}$ is equally plausible to be accepted with the conflicting set $\{e\}$:*

$$\{a\} \equiv^{\tau}_{F_4} \{e\}$$

Next, consider the extension-ranking semantics $\mathsf{LD}^{\mathsf{ad}}$*, which satisfies* cf-soundness *because every most plausible set with respect to* $\mathsf{LD}^{\mathsf{ad}}$ *is admissible and thus conflict-free (for more details see Proposition 3). However, the conflict-free set $\{b\}$ and the conflicting set $\{e\}$ are equally plausible to be accepted*

$$\{b\} \equiv^{\mathsf{LD}^{\mathsf{ad}}}_{F_4} \{e\}$$

Thus, violating respect conflicts.

Finally, let τ be an extension-ranking semantics such that for sets $E, E' \subseteq A$, $E \sqsupset^{\tau}_{F} E'$ if $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$, or if both are conflict-free and $|E| > |E'|$. Otherwise, $E \equiv^{\tau'}_{F} E'$. This extension-ranking semantics satisfies respect conflicts*. For Considering the admissible sets $\{a, c\}$ and $\{a, c, g\}$, we have:*

$$\{a, c, g\} \sqsupset^{\tau'}_{F_4} \{a, c\}$$

However, for cf-completeness*, it must hold that:*

$$\{a, c, g\} \equiv^{\tau'}_{F_4} \{a, c\}$$

The plausibility of accepting a set of arguments should not be affected by arguments that are unconnected to that set. This means that adding or removing unconnected arguments should not influence the plausibility of accepting a set of arguments. A similar concept was proposed by van der Torre and Vesic (2017) through the principles of *Crash-resistance* and *Non-interference* for extension semantics.

When two unconnected abstract argumentation frameworks are combined into one, the relationships between sets in the combined AF should mirror those in the original, smaller AFs. This requirement is formalised by the *composition* principle. Additionally, when an AF is split into unconnected components, the relationship between sets should remain unchanged. This requirement is captured by the *decomposition* principle.

Definition 28. *Let τ be an extension-ranking semantics.*

- *τ satisfies* composition *if for every AF F that can be split into F_1 and F_2 such that $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ and $E, E' \subseteq A_1 \cup A_2$, it holds that*

$$\text{if } \left\{ \begin{array}{c} E \cap A_1 \sqsupseteq^{\tau}_{F_1} E' \cap A_1 \\ E \cap A_2 \sqsupseteq^{\tau}_{F_2} E' \cap A_2 \end{array} \right\} \text{ then } E \sqsupseteq^{\tau}_{F} E'.$$

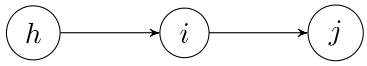

Figure 4.2: AF F_5 from Example 22.

- τ *satisfies* decomposition *if for every AF F that can be split into F_1 and F_2 such that $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ and $E, E' \subseteq A_1 \cup A_2$, it holds that*

$$\text{if } E \sqsupseteq^\tau_F E' \text{ then } \left\{ \begin{array}{c} E \cap A_1 \sqsupseteq^\tau_{F_1} E' \cap A_1 \\ E \cap A_2 \sqsupseteq^\tau_{F_2} E' \cap A_2 \end{array} \right\}.$$

Example 22. *Consider F_4 from Example 6 and $F_5 = (\{h, i, j\}, \{(h, i), (i, j)\})$ as depicted in Figure 4.2.*

Those two abstract argumentation frameworks are disjoint, so for any extension-ranking semantics τ, the relationships in $F_4 \cup F_5$ should be the same as in F_4 and F_5 individually. Consider the sets $\{a, c, g, h, j\}$ and $\{b, c, h, i\}$. The set $\{a, c, g, h, j\}$ is conflict-free in both F_4 and F_5 ($\{a, c, g\} \in \mathsf{cf}(F_4)$ and $\{h, j\} \in \mathsf{cf}(F_5)$), while $\{b, c, h, i\}$ has conflicts in both F_4 and F_5. Therefore, it is reasonable to say that $\{a, c, g, h, j\}$ is more plausible to be accepted than $\{b, c, h, i\}$ in both F_4 and F_5 ($\{b, d\} \notin \mathsf{cf}(F_4)$ and $\{h, i\} \notin \mathsf{cf}(F_5)$):

$$\{a, c, g\} \sqsupset^\tau_{F_4} \{b, c\} \quad \text{and} \quad \{h, j\} \sqsupset^\tau_{F_5} \{h, i\}$$

For composition *to be satisfied, it must hold that:*

$$\{a, c, g, h, j\} \sqsupset^\tau_{F_4 \cup F_5} \{b, c, h, i\}$$

Next, we observe that $\{a, c, g, h, j\}$ is conflict-free and $\{b, c, h, i\}$ is not conflict-free in $F_4 \cup F_5$. Thus, for any extension-ranking semantics τ that generalises $\mathsf{LD}^{\mathsf{cf}}$, we get:

$$\{a, c, g, h, j\} \sqsupset^\tau_{F_4 \cup F_5} \{b, c, h, i\}$$

To satisfy decomposition*, the relationship between $\{a, c, g, h, j\}$ and $\{b, c, h, i\}$ has to remain the same in F_4 and F_5, thus:*

$$\{a, c, g\} \sqsupset^\tau_{F_4} \{b, c\} \quad \text{and} \quad \{h, j\} \sqsupset^\tau_{F_5} \{h, i\}$$

A crucial property of abstract argumentation is *reinstatement*, as discussed by Caminada (2006). Reinstatement involves making an attacked argument acceptable by attacking its attacker. The complete extension semantics implements this property strictly: "if an argument can be reinstated by a set of arguments, it must be included in that set". Thus, a set of arguments that includes an argument it defends is more plausible to be accepted than a set without that argument. We define two versions of this principle: *weak reinstatement* and *strong reinstatement*. The weak version of reinstatement states that adding a defended argument (which does not introduce further conflicts) does not reduce the plausibility of acceptance of any set of arguments. The strong version ensures that adding a defended argument to a set strictly increases the plausibility of acceptance of that set.

Definition 29. *Let τ be an extension-ranking semantics.*

- *τ satisfies* weak reinstatement *if and only if for all AFs $F = (A, R)$ with $E \subseteq A$, it holds that $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$, implies $E \cup \{a\} \sqsupseteq^\tau_F E$.*
- *τ satisfies* strong reinstatement *if and only if for all AFs $F = (A, R)$ with $E \subseteq A$, it holds that $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$, implies $E \cup \{a\} \sqsupset^\tau_F E$.*

Note that the condition $a \notin (E^- \cup E^+)$ is necessary to prevent adding additional conflicts to the set, which could reduce the plausibility of acceptance of the set of arguments.

Example 23. *Consider F_5 from Example 22. The argument j is defended by the argument h. For any extension-ranking semantics τ that satisfies* weak reinstatement, *it must hold that:*

$$\{h, j\} \sqsupseteq^\tau_{F_5} \{h\}$$

For strong reinstatement, the relationship between $\{h, j\}$ and $\{h\}$ must be strict, this means:

$$\{h, j\} \sqsupset^\tau_{F_5} \{h\}$$

If we consider the set $\{h, i\}$, one might think that $\{h, i, j\} \sqsupseteq^\tau_{F_5} \{h, i\}$ should also hold, since j is defended by $\{h, i\}$. However, adding j to the set creates more conflicts, because $j \in \{h, i\}^+_{F_5}$. Therefore, we cannot enforce the relationship between $\{h, i, j\}$ and $\{h, i\}$. In fact, one could argue that $\{h, i\} \sqsupset^\tau_{F_5} \{h, i, j\}$ should hold, as $\{h, i\}$ has fewer conflicts than $\{h, i, j\}$.

Argumentation is inherently non-monotonic, meaning that changes in the abstract argumentation framework can affect the conclusions drawn. However, these changes should align with intuitive reasoning. For instance, adding an attacker to an argument should not strengthen that argument. An extension-ranking semantics should reflect this behaviour, ensuring that if a set E is more plausible to be accepted than another set E', adding attacks from E to E' should not make E less plausible to be accepted than E'. Thus, an extension-ranking semantics should be *robust* to the addition of attacks, which intuitively should not diminish the plausibility of acceptance of a set. A similar robustness concept was discussed by Rienstra et al. (2020) for labelling-based semantics.

Definition 30. *Let τ be an extension-ranking semantics. τ satisfies* addition robustness *if for all AFs $F = (A, R)$ and sets $E, E' \subseteq A$ with $E \sqsupseteq^\tau_F E'$, it holds that $E \sqsupseteq^\tau_{F'} E'$ where $F' = (A, R')$ with $R' = R \cup \{(a, b)\}$ such that $a \in E$, $b \notin E$, and $b \in E'$.*

The principle *addition robustness* ensures that the relationship between two sets is not reversed by adding an attack from a more plausible set to another.

Example 24. *Consider F_4 from Example 6 and the sets $\{a, c, g\}$ and $\{b, c, e\}$. $\{a, c, g\}$ is the stable extension, while $\{b, c, e\}$ is not even conflict-free. Let τ be an arbitrary extension-ranking semantics such that:*

$$\{a, c, g\} \sqsupset^\tau_{F_4} \{b, c, e\}$$

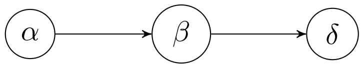

Figure 4.3: AF F_6 from Example 25.

If we create a new AF F_4' by adding an attack from a to e, this provides an additional reason to reject $\{b, c, e\}$. Therefore, if τ satisfies addition robustness*, then:*

$$\{a, c, g\} \sqsupset^{\tau}_{F_4'} \{b, c, e\}$$

In abstract argumentation, the content of each argument is disregarded, focusing solely on the relationships between arguments. It should not matter whether an argument is labelled a or b, as long as the abstract argumentation framework's structure remains the same. Argument-ranking semantics adheres to this requirement through the principle *Abs* (Amgoud and Ben-Naim, 2013)), which states that argument names are irrelevant to the resulting argument ranking. For extension-ranking semantics, we define a principle similar in spirit to *Abs*, called *syntax independence*.

Definition 31. *An extension-ranking semantics τ satisfies* syntax independence *if for every pair of AFs $F = (A, R)$, $F' = (A', R')$, and for every isomorphism $\gamma : A \to A'$, it holds for all $E, E' \subseteq A$, $E \sqsupseteq^{\tau}_{F} E'$ if and only if $\gamma(E) \sqsupseteq^{\tau}_{F'} \gamma(E')$.*

Example 25. *Consider again F_5 from Example 22 and $F_6 = (\{\alpha, \beta, \delta\}, \{(\alpha, \beta), (\beta, \delta)\})$ as depicted in Figure 4.3. It is easy to see that F_6 is isomorphic to F_5 and therefore, the extension rankings with respect to an extension-ranking semantics τ for these two AFs should coincide.*

We conclude the discussion of principles by analysing the least-discriminating extension-ranking semantics from Chapter 3 with respect to the previously defined principles.

The least-discriminating extension-ranking semantics are designed to rank all σ-extensions higher than non-σ-extensions. This implies that the σ-extensions of an AF coincide with the most plausible sets with respect to LD^{σ}. However, LD^{σ} usually does not satisfy σ'*-generalisation* for $\sigma \neq \sigma'$. For example, $\mathsf{LD}^{\mathsf{ad}}$ does not satisfy co*-soundness*.

Example 26. *Consider F_4 from Example 6. The set $\{d\}$ is admissible, implying $\{d\} \in \mathtt{max}_{\mathsf{LD}^{\mathsf{ad}}}(F_4)$, but $\{d\}$ is not a complete set because the argument a is defended by $\{d\}$. Thus, $\mathsf{LD}^{\mathsf{ad}}$ violates* co-soundness. *Similarly, $\mathsf{LD}^{\mathsf{co}}$ does not satisfy* ad-completeness. *The set $\{d\}$ is admissible but not among the most plausible sets with respect to $\mathsf{LD}^{\mathsf{co}}$, violating the requirement that $\{d\}$ should be among the most plausible sets.*

Proposition 3. LD^{σ} *satisfies* σ-generalisation *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

Proof: P. 167

Note that the stable extension semantics is excluded from Proposition 3, as $\mathsf{LD}^{\mathsf{st}}$ violates st*-soundness*. For abstract argumentation frameworks without a stable extension, there should be no most plausible set. However, $\mathsf{LD}^{\mathsf{st}}$ is still defined, though not

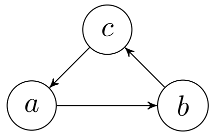

Figure 4.4: AF F_7 from Example 27.

very expressive, as every set is equally plausible to accept and thus among the most plausible sets.

Example 27. *Consider the odd cycle* $F_7 = (\{a, b, c\}, \{(a, b), (b, c), (c, a)\})$*, as depicted in Figure 4.4. This AF has no stable extension. Therefore, for any pair of sets* $E, E' \subseteq \{a, b, c\}$*. Since* $E, E' \notin \mathsf{st}(F_7)$*, it holds that:*

$$E \equiv_{F_7}^{\mathsf{LD}^{\mathsf{st}}} E'$$

Thus, every set is equally plausible to be accepted and considered among the most plausible sets.

In the literature (see Baroni et al. (2018) for an overview), there are more extension semantics σ where there are abstract argumentation frameworks without σ-extensions, thus it is impossible to define an extension-ranking semantics that satisfies *σ-soundness* for these extension semantics.

Proposition 4. *If σ is an extension semantics such that there exists an AF* $F = (A, R)$ *with* $\sigma(F) = \emptyset$*, then no extension-ranking semantics can satisfy* σ-soundness.

Proof: P. 168

We have already shown in Example 21 that $\mathsf{LD}^{\mathsf{ad}}$ violates *respect conflicts*. For $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$, LD^{σ} exhibits the same behaviour.

Example 28. *Consider* F_4 *and the sets* $\{b\}$ *and* $\{e\}$*. Both these sets are not admissible, but* $\{b\}$ *is conflict-free, while* $\{e\}$ *is not. For* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{st}, \mathsf{sst}\}$*, we get:*

$$\{b\} \equiv_{F_4}^{\mathsf{LD}^{\sigma}} \{e\}$$

Therefore, LD^{σ} *violates* respect conflicts *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Only $\mathsf{LD}^{\mathsf{cf}}$ satisfies *respect conflicts*.

Proposition 5. $\mathsf{LD}^{\mathsf{cf}}$ *satisfies* respect conflicts.

Proof: P. 168

When two disjoint abstract argumentation frameworks are combined into a new AF, then acceptance of each set remains unchanged. Specifically, if a set is complete in both original AFs, it remains complete in the combined AF. Therefore, LD^{σ} satisfies *composition* for all considered semantical notions.

Proposition 6. LD^{σ} *satisfies* composition *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof: P. 168

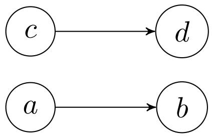

Figure 4.5: AF F_8 from Example 29.

Figure 4.6: AF F_9 from Example 30.

Decomposition is violated by all considered least-discriminating extension-ranking semantics.

Example 29. *Let* $F_8 = (\{a, b, c, d\}, \{(a, b), (c, d)\})$ *be an AF and the two sets* $\{a, b, c\}$ *and* $\{a, c, d\}$*, as depicted in Figure 4.5.*

F_8 *can be partitioned into two disjoint AFs:* $F_{8,1} = (\{a, b\}, \{(a, b)\})$ *and* $F_{8,2} = (\{c, d\}, \{(c, d)\})$*. Both sets are not conflict-free in* F_8*, so for any extension semantics based on conflict-freeness,* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$*, it holds that:*

$$\{a, b, c\} \equiv^{\mathsf{LD}^\sigma}_{F_8} \{a, c, d\}$$

However, $\{a, c, d\} \cap \{a, b\}$ *is conflict-free in* $F_{8,1}$*, and* $\{a, b, c\} \cap \{c, d\}$ *is conflict-free in* $F_{8,2}$*. Thus, we get:*

$$\{a, c, d\} \cap \{a, b\} \sqsupset^{\mathsf{LD}^\sigma}_{F_{8,1}} \{a, b, c\} \cap \{a, b\} \quad \text{and} \quad \{a, b, c\} \cap \{c, d\} \sqsupset^{\mathsf{LD}^\sigma}_{F_{8,2}} \{a, c, d\} \cap \{c, d\}$$

for $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$*. Hence,* LD^σ *violates* decomposition *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Adding an argument that is already defended does not lead to the rejection of the corresponding set with respect to any extension semantics. Thus, LD^σ satisfies *weak reinstatement* for all considered semantical notions.

Proposition 7. LD^σ *satisfies* weak reinstatement *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof: P. 170

While all least-discriminating extension-ranking semantics satisfy weak reinstatement, they violate *strong reinstatement* because they cannot distinguish between two non-acceptable sets. This is demonstrated with a counterexample.

Example 30. *Let* $F_9 = (\{a, b\}, \{(a, a)\})$ *be an AF, as depicted in Figure 4.6, consider* $E = \{a\}$*. Since* E *is not conflict-free,* $E \notin \sigma(F_9)$ *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{st}, \mathsf{sst}\}$*. However,* $b \in \mathfrak{F}_{F_9}(E)$*,* $b \notin E$ *and* $b \notin (E^- \cup E^+)$*. Based on* strong reinstatement*, it would imply*

$$E \cup \{b\} \sqsupset^{\mathsf{LD}^\sigma}_{F_9} E$$

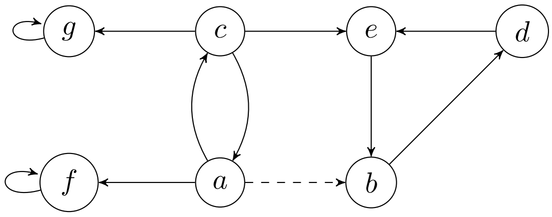

Figure 4.7: AF F_{10} from Example 31, where the dashed attack between a and b, (a, b), is added later to obtain F'_{10}.

but $E \cup \{b\}$ *is not conflict-free. Hence,* $E \cup \{b\} \notin \sigma(F_9)$*, and therefore* $E \cup \{b\} \notin \mathtt{max}_{\mathsf{LD}^\sigma}(F_9)$ *and:*

$$E \cup \{b\} \equiv^{\mathsf{LD}^\sigma}_{F_9} E$$

Thus, LD^σ *does violates* strong reinstatement *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{st}, \mathsf{sst}\}$.

Adding an attack to an already defeated set of arguments does not improve its status. Specifically, if a set is not conflict-free, admissible, grounded, or stable, adding an attack will not make it conflict-free, admissible, grounded, or stable. Therefore, the least-discriminating extension-ranking semantics with respect to conflict-freeness, admissibility, grounded, and stable extension semantics satisfy *addition robustness*.

Proposition 8. LD^σ *satisfies* addition robustness *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{gr}, \mathsf{st}\}$.

Proof: P. 170

For the remaining extension semantics (complete, preferred, and semi-stable), we can give a counterexample showing that *addition robustness* is violated. This is because adding an outgoing attack from a set can result in additional arguments being defended.

Example 31. *Let* F_{10} *be the AF depicted in Figure 4.7. Consider the sets* $\{a\}$ *and* $\{b, c\}$*, both of which are complete, preferred and semi-stable. If we add the attack* (a, b)*, then argument* d *is defended by* $\{a\}$*, making* $\{a\}$ *no longer complete, preferred, and semi-stable. Meanwhile,* $\{b, c\}$ *remains complete, preferred and semi-stable. This shows that:*

$$\{a\} \sqsupseteq^{\mathsf{LD}^\sigma}_{F_{10}} \{b, c\} \quad \textit{but} \quad \{b, c\} \sqsupset^{\mathsf{LD}^\sigma}_{F'_{10}} \{a\}$$

for $\sigma \in \{\mathsf{co}, \mathsf{pr}, \mathsf{sst}\}$*, violating* addition robustness.

Since the least-discriminating extension-ranking semantics only distinguish sets based on their acceptance status, the names of individual arguments are irrelevant. Thus, by definition LD^σ satisfies *syntax independence* for $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{st}, \mathsf{sst}\}$.

The principles satisfied and violated by LD^σ for $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ are summarised in Table 4.1.

In Proposition 2, we have shown that *respect conflicts* implies cf-*soundness*, and in Example 21, we demonstrated that the reverse is not true. It is also evident by definition that *strong reinstatement* implies *weak reinstatement*.

Principles	$\mathsf{LD}^{\mathsf{cf}}$	$\mathsf{LD}^{\mathsf{ad}}$	$\mathsf{LD}^{\mathsf{co}}$	$\mathsf{LD}^{\mathsf{gr}}$	$\mathsf{LD}^{\mathsf{pr}}$	$\mathsf{LD}^{\mathsf{st}}$	$\mathsf{LD}^{\mathsf{sst}}$
σ-generalisation	✓($\sigma = \mathsf{cf}$)	✓($\sigma = \mathsf{ad}$)	✓($\sigma = \mathsf{co}$)	✓($\sigma = \mathsf{gr}$)	✓($\sigma = \mathsf{pr}$)	✗	✓($\sigma = \mathsf{sst}$)
respect conflicts	✓	✗	✗	✗	✗	✗	✗
composition	✓	✓	✓	✓	✓	✓	✓
decomposition	✗	✗	✗	✗	✗	✗	✗
weak reinstatement	✓	✓	✓	✓	✓	✓	✓
strong reinstatement	✗	✗	✗	✗	✗	✗	✗
addition robustness	✓	✓	✗	✓	✗	✓	✗
syntax independence	✓	✓	✓	✓	✓	✓	✓

Table 4.1: Principles satisfied and violated by LD^σ for $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{st}, \mathsf{sst}\}$.

Next, we discuss the incompatibility of certain principles. The principle of ad-*generalisation* enforces that admissible sets are the most plausible with respect to an extension ranking, which contradicts *strong reinstatement*. This principle enforces strict compliance with reinstatement.

Example 32. *Consider AF F_5 from Example 22 and the sets $\{h\}$ and $\{h, j\}$. Both sets are admissible, so for any extension-ranking semantics τ that satisfies* ad-generalisation*, we get:*

$$\{h\} \equiv^\tau_{F_5} \{h, j\}$$

However, j is defended by $\{h\}$ and can be added to $\{h\}$ without creating new conflicts. Thus, strong reinstatement *implies:*

$$\{h, j\} \sqsupset^\tau_{F_5} \{h\}$$

This shows that we cannot satisfies both principles simultaneously.

Example 31 illustrates that *σ-generalisation* for $\sigma \in \{\mathsf{co}, \mathsf{pr}, \mathsf{sst}\}$ is incompatible with *addition robustness*. As the sets $\{a\}$ and $\{b, c\}$ are both complete, preferred and semi-stable, any extension-ranking semantics τ that satisfies *σ-generalisation* implies:

$$\{a\} \equiv^\tau_{F_{10}} \{b, c\}$$

Adding the attack (a, b) to F_{10} results in:

$$\{b, c\} \sqsupset^\tau_{F'_{10}} \{a\}$$

violating *addition robustness*.

For other pairs of principles (*σ-generalisation*, *respect conflicts*, *composition*, *decomposition*, *weak reinstatement*, *strong reinstatement*, *addition robustness*) we show that they are *pairwise independent*, meaning neither principle implies the other, but they are still compatible with each other.

Proposition 9. *The following holds:*

1. *The principle* respect conflicts *implies* cf-soundness.

2. *The principle* strong reinstatement *implies* weak reinstatement.
3. *The principles* ad-generalisation *and* strong reinstatement *are incompatible.*
4. *The principles* σ-generalisation *for* $\sigma \in \{\mathsf{co}, \mathsf{pr}, \mathsf{sst}\}$ *and* addition robustness *are incompatible.*
5. *For the remaining pairs of principles* $p_1, p_2 \in \{\sigma\text{-generalisation}, \text{respect conflicts}, \text{composition}, \text{decomposition}, \text{weak reinstatement}, \text{strong reinstatement}, \text{addition robustness}\}$ *where* $p_1 \neq p_2$*,* p_1 *and* p_2 *are pairwise independent.*

Proof: P. 175

In this chapter, we introduced several principles that a well-behaved extension-ranking semantics should satisfy. We showed that most of these principles are independent of each other. These principles provide a comprehensive overview of the behaviour of extension-ranking semantics. Additionally, we analysed the behaviour of the least-discriminating extension-ranking semantics and found that this family already satisfies many of these principles, serving as a good baseline for how an extension-ranking semantics should behave.

BASE RELATIONS 5

In the following chapters, we will explore different families of extension-ranking semantics. We begin by defining a general framework using *simple* relations between sets of arguments. Each of these relations captures a single aspect of argumentative reasoning. Individually, these relations are not very expressive, but by combining them, we can derive generalisations of various extension semantics. The combination of these base relations is the subject of Chapter 6. In this chapter, we focus on what we call *base relations*.

We begin with set-based relations, where two sets of arguments are compared based on the set of conflicts (Conflicts), the set of undefended arguments (Undef), the set of defended-and-not-contained arguments (Condef), or the set of unattacked arguments (Unatt) that two sets induce. In the second part of this chapter (Section 5.2), we discuss cardinality-based variations of these base relations, along with other variations.

Before introducing the individual base relations, we define the general concept of a *base relation*. Note that a base relation is not a preorder, meaning it is not necessarily reflexive or transitive.

Definition 32. *Let $F = (A, R)$ be an AF. A binary relation τ is called a* base relation *if and only if $\sqsupseteq^\tau_F \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$.*

In other words, a base relation τ represents the relationship between two sets of arguments E and E' within the context of an abstract argumentation framework F. This allows us to determine whether E is at least as plausible to be accepted as E', whether the two sets are equally plausible to be accepted, or if they are incomparable.

Every extension-ranking semantics is, by definition, a base relation, even if the base relation itself is not very expressive, such as LD^σ.

5.1 SET-BASED BASE RELATIONS

In the evaluation of abstract argumentation frameworks, extension semantics implement central notions in a binary fashion. For instance, *conflict-freeness* requires that extensions contain no conflicts. To achieve a more fine-grained evaluation of the

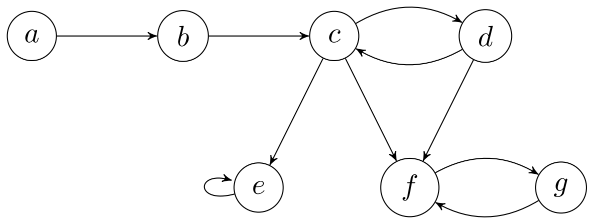

Figure 5.1: Recalled AF F_4 from Example 6.

plausibility of acceptance of sets of arguments, we need a more detailed assessment of these notions. Below, we define base relations that model key aspects of *conflicts, admissibility, completeness,* and *stability* by evaluating how well a given set satisfies each aspect.

The first aspect we consider is *conflicts*, the absence of which is typically seen as a desirable property in any extension semantics. Acceptable sets of arguments should be internally consistent and free from self-contradiction. Specifically, a conflict-free set of arguments is considered more plausible to be accepted than a conflicting set. Extending this idea, we regard a set E to be more plausible to be accepted than another set E' if E has strictly fewer conflicts than E' (in terms of set inclusion). This is modelled by the $\sqsupseteq^{\mathsf{Conflicts}}$ *base relation*.

Definition 33. *Let $F = (A, R)$ be an AF and $E \subseteq A$. Define* Conflicts *via:*

$$\mathsf{Conflicts}_F(E) = \{(a, b) \in R | a, b \in E\}$$

The corresponding Conflicts base relation $\sqsupseteq_F^{\mathsf{Conflicts}}$ *via:*

$$E \sqsupseteq_F^{\mathsf{Conflicts}} E' \textit{ if and only if } \mathsf{Conflicts}_F(E) \subseteq \mathsf{Conflicts}_F(E')$$

Example 33. *Consider F_4 from Example 6 (depicted again in Figure 5.1). Let the sets be $\{b, c\}$ and $\{b, c, d\}$. We have $\mathsf{Conflicts}_{F_4}(\{b, c\}) = \{(b, c)\}$ and $\mathsf{Conflicts}_{F_4}(\{b, c, d\}) = \{(b, c), (c, d), (d, c)\}$. Thus, $\{b, c\}$ is more plausible to be accepted than $\{b, c, d\}$ in terms of conflicts:*

$$\{b, c\} \sqsupset_{F_4}^{\mathsf{Conflicts}} \{b, c, d\}$$

Another generally desirable property is *admissibility*. Sets of arguments should defend themselves against any threat. Specifically, a set of arguments that defends all its elements is considered more plausible to be accepted than a set containing at least one undefended argument. Thus, a set E is more plausible to be accepted than another set E' if E has strictly fewer undefended arguments than E' (in terms of set inclusions). This is captured by the $\sqsupseteq^{\mathsf{Undef}}$ *base relation* (Undef stands for "undefended"), where $\mathfrak{F}$ is the characteristic function.

Definition 34. *Let $F = (A, R)$ be an AF and $E \subseteq A$. Define* Undef *via:*

$$\mathsf{Undef}_F(E) = E \setminus \mathfrak{F}_F(E)$$

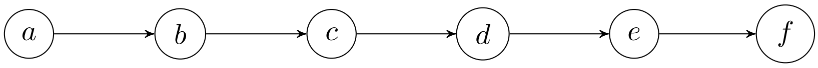

Figure 5.2: AF F_{11} from Example 35.

The corresponding Undef base relation $\sqsupseteq_F^{\mathsf{Undef}}$ *via:*

$$E \sqsupseteq_F^{\mathsf{Undef}} E' \text{ if and only if } \mathsf{Undef}_F(E) \subseteq \mathsf{Undef}_F(E')$$

Example 34. *Consider F_4 from Example 6 and the sets $\{b\}$ and $\{b, f\}$. Both arguments b and f are not defended by $\{b\}$ and $\{b, f\}$, respectively. However $\{b\}$ contains fewer undefended arguments, so:*

$$\{b\} \sqsupset_{F_4}^{\mathsf{Undef}} \{b, f\}$$

A complete extension includes every defended argument. Thus, a set that contains all its defended arguments (without adding conflicts) is more plausible to be accepted than a set that does not include all the arguments it defends. More generally, a set E is more plausible to be accepted than another set E' if there are fewer arguments *consistently defended* by E and not contained in E than there are for E' (in terms of set inclusion). By *consistent defence*, we mean arguments that are defended by E and do not attack E or are not attacked by E. To adequately model this notion, we need a more general notion of defence than that provided by the characteristic function $\mathfrak{F}_F$.

Example 35. *Consider AF F_{11} depicted in Figure 5.2 and the set $\{b\}$. We ask:*

What arguments are defended by $\{b\}$?

Observe that $\mathfrak{F}_{F_{11}}(\{b\}) = \{a, d\}$. Argument d is only defended because of b, but argument a is also defended by $\{b\}$, even though it attacks b. If we look at the defended arguments of $\{a, d\}$, we get $\mathfrak{F}_{F_{11}}(\{a, d\}) = \{a, c, f\}$. The defence of argument d is lost, although it was defended before, and again we have an argument f whose defence depends on the acceptance of b, but this argument is not part of the set of defended arguments. The fixed point of this procedure is the grounded extension, $\{a, c, e\}$.

Example 35 illustrates the need for a different definition of the characteristic function: any incoming attack on the set under examination has to be ignored. In the example, the attack from argument a has to be ignored, and then we compute the defended arguments, assuming that the set $\{b\}$ is acceptable. The expected result is then $\{b, d\}$. If we iterate this notion of consistent defence, we end up with the set $\{b, d, f\}$. This set models the intuition that d and f are consistently defended if we assume that $\{b\}$ is accepted. This function is designed as a generalisation of Dung's characteristic function $\mathfrak{F}^*$.

Definition 35. *The function $\mathfrak{F}_F^* : \mathcal{P}(A) \to \mathcal{P}(A)$ is defined via:*

$$\mathfrak{F}_F^*(E) = \bigcup_{i=1}^{\infty} \mathfrak{F}_{i,F}^*(E)$$

with

$$\begin{aligned}\mathfrak{F}^*_{1,F}(E) &= E\\ \mathfrak{F}^*_{i,F}(E) &= \mathfrak{F}^*_{i-1,F}(E) \cup (\mathfrak{F}_F(\mathfrak{F}^*_{i-1,F}(E)) \setminus E^-)\end{aligned}$$

With $\mathfrak{F}^*$, we get our desired results in Example 35, i. e. $\mathfrak{F}^*_{F_{11}}(\{b\}) = \{b, d, f\}$.

To describe $\mathfrak{F}^*$ in other words: We start by assuming that the set to be investigated, E, is admissible. Then every defended argument by E is added, and every attacker of E is removed. Defended arguments are added until a fixed point is reached.

Proposition 10. *Let $F = (A, R)$ be an AF and $E \subseteq A$. Then there exists a $k \in \mathbb{N}$ with:*

$$E \subseteq \mathfrak{F}^*_{1,F}(E) \subseteq \mathfrak{F}^*_{2,F}(E) \subseteq \cdots \subseteq \mathfrak{F}^*_{k-1,F}(E) \subseteq \mathfrak{F}^*_{k,F}(E) = \mathfrak{F}^*_{k+1,F}(E) = \ldots$$

Proof: P. 179

Next, we discuss $\mathfrak{F}^*$ further and analyse its behaviour. $\mathfrak{F}^*$ is supposed to be a generalisation of Dung's characteristic function, so these two functions should give the same results if the input is admissible.

Proposition 11. *Let $F = (A, R)$ be an AF, $E \subseteq A$ such that $E \in \mathsf{ad}(F)$. Then $\mathfrak{F}_F(E) = \mathfrak{F}^*_F(E)$.* *Proof: P. 180*

A key property of the characteristic function $\mathfrak{F}_F$ is that the grounded extension is the least fixed point of that function.

Proposition 12. *Let $F = (A, R)$ be an AF and $E = \mathsf{gr}(F)$, then E is the least fixed point of $\mathfrak{F}^*_F$.* *Proof: P. 180*

Using Proposition 11, we can show that every complete extension of an abstract argumentation framework F is also a fixed point of $\mathfrak{F}^*_F$.

Proposition 13. *Let $F = (A, R)$ be an AF and $E \in \mathsf{co}(F)$. Then $\mathfrak{F}^*_F(E) = E$.*

Proof: P. 180

To conclude this discussion on $\mathfrak{F}^*$, we provide a detailed example to illustrate its behaviour more thoroughly.

Example 36. *Consider F_4 from Example 6 and the set $\{b\}$. Argument b is attacked by argument a, so $b^-_{F_4} = \{a\}$. To compute all consistently defended arguments by $\{b\}$, we use $\mathfrak{F}^*_{F_4}(\{b\})$. By definition, we have $\mathfrak{F}^*_{1,F_4}(\{b\}) = \{b\}$. Since arguments a and d are defended by $\{b\}$ and argument b is not defended, we have $\mathfrak{F}_{F_4}(\{b\}) = \{a, d\}$ and therefore $\mathfrak{F}^*_{2,F_4}(\{b\}) = \{b\} \cup (\{a, d\} \setminus \{a\}) = \{b, d\}$. For the next iteration of $\mathfrak{F}^*_{F_4}$, we see that a, d and g are defended by $\{b, d\}$, $\mathfrak{F}_{F_4}(\{b, d\}) = \{a, d, g\}$. Therefore, $\mathfrak{F}^*_{3,F_4}(\{b\}) = \{b, d\} \cup (\{a, d, g\} \setminus \{a\}) = \{b, d, g\}$. Since $\{b, d, g\}$ also defends arguments a, d, and g, we reach a fixed point, $\mathfrak{F}^*_{4,F_4}(\{b\}) = \{b, d, g\}$. Combining each iteration, we get:*

$$\mathfrak{F}^*_{F_4}(\{b\}) = \{b, d, g\}$$

Finally, we define the $\sqsupseteq^{\mathsf{Condef}}$ *base relation*, which models the concept of *consistent defence* (Condef meaning "consistently defended and not in").

Definition 36. *Let $F = (A, R)$ be an AF and $E \subseteq A$. We define* Condef *via:*

$$\mathsf{Condef}_F(E) = \mathfrak{F}^*_F(E) \setminus E$$

The corresponding Condef base relation $\sqsupseteq^{\mathsf{Condef}}_F$ *via:*

$$E \sqsupseteq^{\mathsf{Condef}}_F E' \text{ if and only if } \mathsf{Condef}_F(E) \subseteq \mathsf{Condef}_F(E').$$

Example 37. *Continuing with Example 36, using F_4 from Example 6, we have already calculated $\mathfrak{F}^*_{F_4}(\{b\}) = \{b, d, g\}$, so using* Condef, *we get* $\mathsf{Condef}_{F_4}(\{b\}) = \{d, g\}$. *Next, we examine the set $\{b, d\}$, where $\mathfrak{F}^*_{F_4}(\{b, d\}) = \{b, d, g\}$, so $\mathsf{Condef}_{F_4}(\{b, d\}) = \{g\}$. Therefore, we have:*

$$\{b, d\} \sqsupset^{\mathsf{Condef}}_{F_4} \{b\}$$

The final property we examine is *stability*. A set is considered stable if it attacks every argument not contained within it. To generalise this concept, we assert that a set E is more plausible to be accepted than another set E' if E strictly attacks more arguments than E' (in terms of set inclusion). We capture this aspect with the $\sqsupseteq^{\mathsf{Unatt}}$ *base relation* (Unatt stands for "unattacked").

Definition 37. *Let $F = (A, R)$ be an AF and $E \subseteq A$. Define* Unatt *via:*

$$\mathsf{Unatt}_F(E) = A \setminus (E \cup E^+_F)$$

The corresponding Unatt base relation $\sqsupseteq^{\mathsf{Unatt}}_F$ *via:*

$$E \sqsupseteq^{\mathsf{Unatt}}_F E' \text{ if and only if } \mathsf{Unatt}_F(E) \subseteq \mathsf{Unatt}_F(E')$$

Example 38. *Consider F_4 from Example 6 and the two sets $\{a, c\}$ and $\{a, d\}$. Argument b is attacked by a, and f is attacked by both c and d, while argument g is not attacked by either set. Argument e is only attacked by c. To summarise:*

$$\mathsf{Unatt}_{F_4}(\{a, c\}) = \{g\} \quad \textit{and} \quad \mathsf{Unatt}_{F_4}(\{a, d\}) = \{e, g\}$$

This shows that :

$$\{a, c\} \sqsupset^{\mathsf{Unatt}}_{F_4} \{a, d\}$$

We will now delve deeper into the behaviour of the base relations. We will demonstrate that these base relations are both reflexive and transitive.

Proposition 14. *The base relation $\sqsupseteq^{\tau}_F$ is reflexive and transitive for $\tau \in \{$*Conflicts, Undef, Condef, Unatt$\}$. *Proof: P. 181*

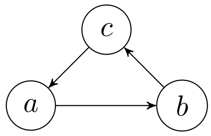

Figure 5.3: Recalled AF F_7 from Example 27.

The base relations $\sqsupseteq^\tau_F$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ induce extension rankings. We are interested in investigating the satisfied and violated principled of the induced extension rankings.

We begin by showing that the set containing all arguments can be among the most plausible sets for $\tau \in \{\mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.

Example 39. *Consider AF F_7 from Example 27, recalled in Figure 5.3. The set $\{a, b, c\}$ defends all its elements and contains all argument, so:*

$$\mathsf{Undef}_{F_7}(\{a, b, c\}) = \mathsf{Condef}_{F_7}(\{a, b, c\}) = \mathsf{Unatt}_{F_7}(\{a, b, c\}) = \emptyset$$

Therefore, there cannot be a set strictly more plausible to be accepted than $\{a, b, c\}$ with respect to $\sqsupseteq^\tau$ for $\tau \in \{\mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, implying $\{a, b, c\} \in \mathtt{max}_{\sqsupseteq^\tau}(F_7)$.

Example 39 directly shows that *σ-generalisation* and *respect conflicts* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ are violated by $\sqsupseteq^\tau$ with $\tau \in \{\mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, since as soon as an abstract argumentation framework has an attack, the set A is not conflict-free, even though it is one of the most plausible sets.

For Conflicts, we show that all conflict-free sets are equally plausible to be accepted. This implies that *respect conflicts* is satisfied, but it also implies that *σ-generalisation* is violated by $\sqsupseteq^{\mathsf{Conflicts}}$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proposition 15. *Let $F = (A, R)$ be an AF. Then $\mathsf{cf}(F) = \mathtt{max}_{\sqsupseteq^{\mathsf{Conflicts}}}(F)$.* *Proof: P. 181*

Example 40. *Consider AF F_4 from Example 6 and the sets $\{a\}$ and $\{b\}$. The set $\{a\}$ is admissible, while $\{b\}$ is not admissible. However, both sets are conflict-free, implying:*

$$\{a\} \equiv^{\mathsf{Conflicts}}_{F_4} \{b\}$$

*Therefore, σ-*generalisation *for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ is violated by $\sqsupseteq^{\mathsf{Conflicts}}$.*

The following lemma will be useful to prove (de)composition.

Lemma 2. *For $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, if AFs $F_1 = (A_1, R_1)$, $F_2 = (A_2, R_2)$ and $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ then $\tau_{F_1}(E \cap A_1) \cup \tau_{F_2}(E \cap A_2) = \tau_F(E)$ for every $E \subseteq A_1 \cup A_2$.* *Proof: P. 181*

Next, we show that the base relations satisfy *composition* and *decomposition*.

Proposition 16. *The base relation $\sqsupseteq^\tau$ satisfies* composition *and* decomposition *for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.* *Proof: P. 181*

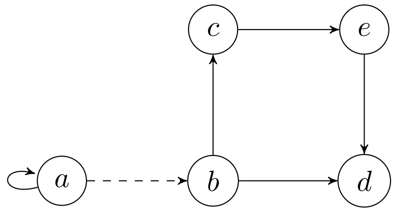

Figure 5.4: AF F_{12} from Example 41, where the attack (a, b) is added later to obtain F'_{12}.

We now investigate the two reinstatement principles.

Proposition 17. $\sqsupseteq^\tau$ *satisfies* strong reinstatement *for* $\tau \in \{\mathsf{Condef}, \mathsf{Unatt}\}$ *and* weak reinstatement *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}\}$. *Proof: P. 182*

In the proof for Proposition 17, we showed that $E \cup \{a\} \equiv^\tau_F E$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}\}$, so *strong reinstatement* is violated.

The principle *addition robustness* is satisfied by $\sqsupseteq^\tau$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Unatt}\}$.

Proposition 18. $\sqsupseteq^\tau$ *satisfies* addition robustness *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Unatt}\}$.
Proof: P. 182

For $\sqsupseteq^{\mathsf{Undef}}$ and $\sqsupseteq^{\mathsf{Condef}}$, we show that *addition robustness* is violated.

Example 41. *Consider AF* $F_{12} = (\{a, b, c, d, e\}, \{(a, a), (b, c), (b, d), (c, e), (e, d)\})$ *as depicted in Figure 5.4 and the sets* $\{a, d\}$ *and* $\{a, b, c, d\}$*. Argument* d *is undefended by both sets, and additionally* $\{a, b, c, d\}$ *does not defend argument* c*, so* $\mathsf{Undef}_{F_{12}}(\{a, d\}) = \{d\}$ *and* $\mathsf{Undef}_{F_{12}}(\{a, b, c, d\}) = \{c, d\}$ *and therefore:*

$$\{a, d\} \sqsupset^{\mathsf{Undef}}_{F_{12}} \{a, b, c, d\}$$

We add the attack (a, b) *into* F_{12} *to obtain* F'_{12}*. However, in* F'_{12} *the set* $\{a, b, c, d\}$ *defends all its elements, i. e.,* $\mathsf{Undef}_{F'_{12}}(\{a, b, c, d\}) = \emptyset$*, while argument* d *is still not defended by* $\{a, d\}$*. This implies:*

$$\{a, b, c, d\} \sqsupset^{\mathsf{Undef}}_{F'_{12}} \{a, d\}$$

Therefore $\sqsupseteq^{\mathsf{Undef}}$ *violates* addition robustness.

For $\sqsupseteq^{\mathsf{Condef}}$*, we get* $\mathsf{Condef}_{F_{12}}(\{a, d\}) = \mathsf{Condef}_{F_{12}}(\{a, b, c, d\}) = \emptyset$*. We can add the attack* (a, b) *and create* F'_{12}*. However, in* F'_{12}*, argument* c *is consistently defended by* $\{a, d\}$*, hence* $\mathsf{Condef}_{F'_{12}}(\{a, d\}) = \{c\}$*. For set* $\{a, b, c, d\}$*,* $\mathsf{Condef}_{F'_{12}}(\{a, b, c, d\}) = \emptyset$*, implying:*

$$\{a, b, c, d\} \sqsupset^{\mathsf{Condef}}_{F'_{12}} \{a, d\}$$

Therefore addition robustness *is violated.*

By definition, the base relations $\sqsupseteq^\tau$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ satisfy *syntax independence*, since these base relations are not influenced by the names of the arguments.

We summarise the satisfied and violated principles of $\sqsupseteq^\tau$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ in Table 5.1.

Principles	$\sqsupseteq^{\mathsf{Conflicts}}$	$\sqsupseteq^{\mathsf{Undef}}$	$\sqsupseteq^{\mathsf{Condef}}$	$\sqsupseteq^{\mathsf{Unatt}}$
σ-generalisation	✗	✗	✗	✗
respect conflicts	✓	✗	✗	✗
composition	✓	✓	✓	✓
decomposition	✓	✓	✓	✓
weak reinstatement	✓	✓	✓	✓
strong reinstatement	✗	✗	✓	✓
addition robustness	✓	✗	✗	✓
syntax independence	✓	✓	✓	✓

Table 5.1: Principles satisfied and violated by $\sqsupseteq^{\tau}$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.

5.2 CARDINALITY-BASED BASE RELATIONS

So far we have used subset comparisons to compare sets of arguments with the base relations. However, other comparison methods are imaginable. A typical alternative to subset comparisons is *cardinality*. Thus, instead of comparing two sets based on set-inclusion, we compare them by the amount of elements these sets contain. For example, c-Conflicts compares two sets based on the number of conflicts. To avoid repeating each definition in detail, we define a generalised cardinality-based base relation $\sqsupseteq^{\mathsf{c}\text{-}\tau}$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.

Definition 38. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. We define* c-τ *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ *as the* cardinality-based base relation $\sqsupseteq_F^{\mathsf{c}\text{-}\tau}$ *via:*

$$E \sqsupseteq_F^{\mathsf{c}\text{-}\tau} E' \text{ if and only if } |\tau_F(E)| \leq |\tau_F(E')|$$

Example 42. *Consider AF F_4 from Example 6 and the sets $\{c, g\}$ and $\{b, f\}$. Argument g is defended by $\{c, g\}$, while c is not defended, and both b and f are not defended by $\{b, f\}$. These two sets are incomparable with respect to* Undef*, because $\{c\}$ is not a subset of $\{b, f\}$. However, using the cardinality-based version* c-Undef*, these two sets can be compared, and we get:*

$$\{c, g\} \sqsupseteq_{F_4}^{\mathsf{c}\text{-}\mathsf{Undef}} \{b, f\}$$

Next, we discuss the relationship between the cardinality-based base relations and the subset-based ones. We show that the cardinality variants c-τ are generalisations of τ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.

Proposition 19. *For AF $F = (A, R)$, sets $E, E' \subseteq A$, and $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, it holds that if $E \sqsupseteq_F^{\tau} E'$ then $E \sqsupseteq_F^{\mathsf{c}\text{-}\tau} E'$.* *Proof: P. 182*

In other words, if the subset variant of a base relation can determine the relationship between two sets of arguments, then the subset variant will give us the same result. In Example 42, we saw that two sets which are incomparable with respect to $\sqsupseteq_{F_4}^{\mathsf{Undef}}$ are comparable with respect to $\sqsupseteq_{F_4}^{\mathsf{c}\text{-}\mathsf{Undef}}$.

Proposition 19 directly shows that $\sqsupseteq^{\text{c-}\tau}$ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ is reflexive and transitive, since $\sqsupseteq^{\tau}$ is reflexive and transitive as shown in Proposition 14. Consequently, $\sqsupseteq^{\text{c-}\tau}$ induces extension rankings.

Next, we examine the principles of the induced extension ranking. Since $\sqsupseteq^{\text{c-}\tau}$ is a generalisation of $\sqsupseteq^{\tau}$, we can use the same examples to show that $\sqsupseteq^{\text{c-}\tau}$ violates certain principles. Example 39 shows that $\sqsupseteq^{\text{c-}\tau}$ for $\tau \in \{\mathsf{Unatt}, \mathsf{Condef}, \mathsf{Unatt}\}$ violates *σ-generalisation* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ and *respect conflicts*. $\sqsupseteq^{\text{c-Conflicts}}$ violates *σ-generalisation* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ because of Example 40. Example 41 shows that *addition robustness* is violated by $\sqsupseteq^{\text{c-}\tau}$ for $\tau \in \{\mathsf{Undef}, \mathsf{Condef}\}$.

For the remaining principles, we can base our proofs on those for the subset variant. We start with *respect conflicts* for $\sqsupseteq^{\text{c-Conflicts}}$.

Proposition 20. $\sqsupseteq^{\text{c-Conflicts}}$ *satisfies* respect conflicts. *Proof: P. 183*

To show the satisfaction of *composition*, we show a similar lemma to Lemma 2.

Lemma 3. *For* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$*, if* $F_1 = (A_1, R_1)$*,* $F_2 = (A_2, R_2)$ *and* $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ *with* $A_1 \cap A_2 = \emptyset$ *then* $|\tau_{F_1}(E \cap A_1)| + |\tau_{F_2}(E \cap A_2)| = |\tau_F(E)|$ *for every* $E \subseteq A_1 \cup A_2$*.* *Proof: P. 183*

Using Lemma 3, we can show that the cardinality variants of the base relations satisfy *composition*.

Proposition 21. $\sqsupseteq^{\text{c-}\tau}$ *satisfies* composition *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$. *Proof: P. 183*

The difference between the two variants of base relations is that $\sqsupseteq^{\text{c-}\tau}$ violates *decomposition* for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, while $\sqsupseteq^{\tau}$ satisfies this principle. Thus, we trade totality with *decomposition*.

Example 43. *Consider AF* $F_{13} = (\{a, b, c, d, e, f, g, h\}, \{(a, b), (a, c), (b, d), (c, e), (f, g), (g, h)\})$ *as depicted in Figure 5.5 and* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$*. This AF can be partitioned into two disjoint AFs* $F_{13,1} = (\{a, b, c, d, e\}, \{(a, b), (a, c), (b, d), (c, e)\})$ *and* $F_{13,2} = (\{f, g, h\}, \{(f, g),$
$(g, h)\})$*. Suppose sets* $\{a, b, c\}$ *and* $\{f, g\}$*, then* $|\tau_{F_{13}}(\{a, b, c\})| = 2$ *and* $|\tau_{F_{13}}(\{f, g\})| = 1$*, for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$*. However, we also get:*

$$|\mathsf{Conflicts}_{F_{13,1}}(\{a, b, c\} \cap \{a, b, c, d, e\})| = 2 \quad |\mathsf{Conflicts}_{F_{13,2}}(\{a, b, c\} \cap \{f, g, h\})| = 0$$
$$|\mathsf{Conflicts}_{F_{13,1}}(\{f, g\} \cap \{a, b, c, d, e\})| = 0 \quad |\mathsf{Conflicts}_{F_{13,2}}(\{f, g\} \cap \{f, g, h\})| = 1$$

Thus, we get:

$$\{f, g\} \sqsupset^{\text{c-Conflicts}}_{F_{13}} \{a, b, c\} \quad \textit{but} \quad \{a, b, c\} \cap \{f, g, h\} \sqsupset^{\text{c-Conflicts}}_{F_{13,2}} \{f, g\}$$

Next, consider the sets $\{g\}$ *and* $\{b, c\}$*. For these two sets, we get:*

$$\{g\} \sqsupset^{\text{c-Undef}}_{F_{13}} \{b, c\} \quad \textit{but} \quad \{b, c\} \cap \{f, g, h\} \sqsupset^{\text{c-Undef}}_{F_{13,2}} \{g\} \cap \{f, g, h\}$$

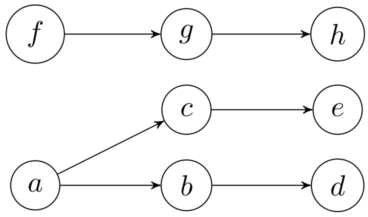

Figure 5.5: AF F_{13} from Example 43.

For sets $\{f\}$ and $\{a\}$, we get:

$$\begin{aligned}
&\{f\} \equiv^{\textsf{c-Condef}}_{F_{13}} \{a\} \textit{ but}\\
&\{a\} \cap \{a,b,c,d,e\} \sqsupset^{\textsf{c-Undef}}_{F_{13,1}} \{f\} \cap \{a,b,c,d,e\} \textit{ and}\\
&\{f\} \cap \{f,g,h\} \sqsupset^{\textsf{c-Undef}}_{F_{13,2}} \{a\} \cap \{f,g,h\}
\end{aligned}$$

Consider the sets $\{f\}$ and $\{a\}$, we have:

$$\{a\} \sqsupset^{\textsf{c-Unatt}}_{F_{13}} \{f\} \textit{ and } \{f\} \cap \{f,g,h\} \sqsupset^{\textsf{c-Unatt}}_{F_{13,2}} \{a\} \cap \{f,g,h\}$$

Thus, for $\tau \in \{\textsf{Conflicts}, \textsf{Undef}, \textsf{Condef}, \textsf{Unatt}\}$, $\sqsupseteq^{\textsf{c-}\tau}$ violates decomposition.

For the reinstatement principles, we base our proofs on the ones for the subset variants of τ.

Proposition 22. $\sqsupseteq^{\textsf{c-}\tau}$ *satisfies* strong reinstatement *for $\tau \in \{\textsf{Condef}, \textsf{Unatt}\}$ and* weak reinstatement *for $\tau \in \{\textsf{Conflicts}, \textsf{Undef}\}$.* *Proof: P. 183*

We have already argued that $\sqsupseteq^{\textsf{c-}\tau}$ for $\tau \in \{\textsf{Undef}, \textsf{Condef}\}$ violates *addition robustness*. We now show that for the other two base relations *addition robustness* is satisfied.

Proposition 23. $\sqsupseteq^{\textsf{c-}\tau}$ *satisfies* addition robustness *for $\tau \in \{\textsf{Conflicts}, \textsf{Unatt}\}$.*
Proof: P. 183

Since the base relations $\tau \in \{\textsf{Conflicts}, \textsf{Undef}, \textsf{Condef}, \textsf{Unatt}\}$ are not influenced by the names of the arguments, it is clear by definition that *syntax independence* is satisfied by $\sqsupseteq^{\textsf{c-}\tau}$.

The satisfied and violated principles of $\sqsupseteq^{\textsf{c-}\tau}$ for $\tau \in \{\textsf{Conflicts}, \textsf{Undef}, \textsf{Condef}, \textsf{Unatt}\}$ are summarised in Table 5.2.

Konieczny et al. (2015) use comparisons of extensions to restrict the credulous or skeptical acceptance of arguments to only the "best" extensions. We extend their definitions to apply to arbitrary pairs of sets of arguments, not just σ-extensions, and define new base relations. First, we recall the concept of *strong defence*. The standard notion of defence suggests that arguments can defend themselves against their attackers. However, this self-defence has been criticised by Baroni and Giacomin (2007), leading to the proposal of a stronger notion called *strong defence*.

Principles	$\sqsupseteq^{\text{c-Conflicts}}$	$\sqsupseteq^{\text{c-Undef}}$	$\sqsupseteq^{\text{c-Condef}}$	$\sqsupseteq^{\text{c-Unatt}}$
σ-generalisation	✗	✗	✗	✗
respect conflicts	✓	✗	✗	✗
composition	✓	✓	✓	✓
decomposition	✗	✗	✗	✗
weak reinstatement	✓	✓	✓	✓
strong reinstatement	✗	✗	✓	✓
addition robustness	✓	✗	✗	✓
syntax independence	✓	✓	✓	✓

Table 5.2: Principles satisfied and violated by $\sqsupseteq^{\text{c-}\tau}$ for $\tau \in \{\text{Conflicts}, \text{Undef}, \text{Condef}, \text{Unatt}\}$.

Definition 39. *Let $F = (A, R)$ be an AF, $a \in A$, and $E \subseteq A$. An argument a is* strongly defended *by E (denoted as $sd_F(a, E, E')$) from $E' \subseteq A$ if and only if for every argument $b \in E'$ with $(b, a) \in R$, there is an argument $c \in E \setminus \{a\}$ such that $(c, b) \in R$ and $sd_F(c, E \setminus \{a\}, E')$.*

In other words, an argument a is *strongly defended* by a set E if E contains a defender of a, which is not a, for every attacker of a, and each defender c must be defended by E without including a and itself. By $sd_F(E, E')$, we refer to the strongly defended arguments by E from E' in abstract argumentation framework F. Using the notion of strong defence, extension semantics like *strongly preferred semantics* can be defined similar to Definition 9 (Baroni and Giacomin, 2007; Caminada, 2014).

Next, we define two base relations based on notions of Konieczny et al. (2015).

Definition 40. *Let $F = (A, R)$ be an AF, and $E, E' \subseteq A$ be two sets of arguments. We define $\tau \in \{\text{nonatt}, \text{strdef}\}$ as base relations τ as follows:*

- $E \sqsupseteq_F^{\text{nonatt}} E'$ *if* $|E \setminus (E_F^- \cap E')| \geq |E' \setminus (E_F'^- \cap E)|$.
- $E \sqsupseteq_F^{\text{strdef}} E'$ *if* $|sd_F(E, E')| \geq |sd_F(E', E)|$.

Intuitively, sets with more unattacked arguments are more plausible to be accepted. The nonatt relation models this idea directly for a pair of sets of arguments. If a set E is attacked less by E' than vice versa, it means E faces fewer threats from E' than vice versa, making E more plausible to accept than E'. The strdef relation extends this idea by requiring strong defence instead of merely considering unattacked arguments.

Example 44. *Consider F_4 from Example 6. The sets $\{a, c, g\}$ and $\{a, d\}$ are both admissible sets. $\{a, c, g\}$ attacks the argument d, while $\{a, d\}$ only attacks c. This means $\{a, c, g\}$ contains more unattacked arguments than $\{a, d\}$, leading to:*

$$\{a, c, g\} \sqsupset_{F_4}^{\text{nonatt}} \{a, d\}$$

For strdef*, consider the sets $\{a, b, f\}$ and $\{g\}$. $\{g\}$ is an admissible set, while $\{a, b, f\}$ is not conflict-free. However, argument a is strongly defended by $\{a, b, f\}$ against $\{g\}$,* *whereas g is not strongly defended by $\{g\}$ against the attack of argument f. Thus, the number of strongly defended arguments in $\{a, b, f\}$ is higher than in $\{g\}$, resulting in:*

$$\{a, b, f\} \sqsupset^{\mathsf{strdef}}_{F_4} \{g\}$$

In Example 44, we see that $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$ behave differently from $\sqsupseteq^{\mathsf{Undef}}$. For $\sqsupseteq^{\mathsf{Undef}}$, we have $\{a, c, g\} \equiv^{\mathsf{Undef}}_{F_4} \{a, d\}$ and $\{g\} \sqsupset^{\mathsf{Undef}}_{F_4} \{a, b, f\}$.

Note that $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$ are not transitive, so these two base relations do not inherently induce extension rankings.

Example 45. *Recall $F_7 = (\{a, b, c\}, \{(a, b), (b, c), (c, a)\})$ from Example 27. Consider the sets $\{a\}$, $\{b\}$ and $\{c\}$. If we compare these sets with* nonatt*, we get:*

$$\{a\} \sqsupset^{\mathsf{nonatt}}_{F_7} \{b\} \quad \textit{and} \quad \{b\} \sqsupset^{\mathsf{nonatt}}_{F_7} \{c\}$$

Assuming transitivity, it must hold:

$$\{a\} \sqsupset^{\mathsf{nonatt}}_{F_7} \{c\}$$

However, this is false because c is not attacked by a, while c attacks a, leading to:

$$\{c\} \sqsupset^{\mathsf{nonatt}}_{F_7} \{a\}$$

This violates transitivity, and the same behavior applies to $\sqsupseteq^{\mathsf{strdef}}_{F_7}$.

Besides violating transitivity, $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$ have another drawback: they ignore internal conflicts within sets, making it irrelevant whether a set is conflict-free or not. Even worse, the set containing every argument A is ranked “better” than any other set.

Proposition 24. *For AF $F = (A, R)$, $A \sqsupseteq^{\tau}_{F} E$ for any $E \subset A$ with $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$.*

Proof: P. 184

The set A is always among the most conflicting sets (for any AF with attacks) and should therefore be among the least acceptable sets, as conflict-freeness is a desirable property in formal argumentation. All σ-extensions according to Dung’s extension semantics are conflict-free. Other extension semantics, such as *naive*, *stage* or *CF2* (Baroni et al., 2018; Verheij, 1996, 2003; Baroni and Giacomin, 2003; Baroni et al., 2005) are also based on conflict-freeness.

We have not explored a subset variant of the $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$, since these variants do not align with their underlying motivations.

Example 46. *Consider AF F_4 from Example 6 and the sets $\{a, c\}$ and $\{a, g\}$. Neither set attacks the other, so $\{a, c\} \setminus (\{a, c\}^-_F \cap \{a, g\}) = \{a, c\}$ and $\{a, g\} \setminus (\{a, g\}^-_F \cap \{a, c\}) = \{a, g\}$. With the base relation* nonatt*, we can compare these sets. Despite sharing on argument (a), we cannot compare them using subset comparison because they are not subsets of each other.*

Example 46 highlights the limitations of a subset variation of $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$. Although we can easily identify the unattacked arguments in each set, the sets can still be incomparable.

In this chapter, we have introduced the notion of base relations to compare two sets of arguments. Each base relation models a simple aspect of argumentative reasoning within an abstract argumentation framework, whether it is comparing sets based on their conflicts, undefended arguments, consistently defended arguments, or unattacked arguments.

COMBINATION-BASED APPROACHES 6

The base relations from the previous chapter focus on individual aspects of argumentative reasoning, such as minimising conflicts or maximising self-defence. However, these base relations alone do not fully capture extension-based reasoning for abstract argumentation frameworks. To comprehensively model argumentative reasoning, we need to combine the base relations.

Example 47. *Recall F_5 from Example 22 (depicted again in Figure 6.1). Since $\{j\}$ is conflict-free and $\{h,i,j\}$ is not, we argued that $\{j\}$ is more plausible to be accepted than $\{h,i,j\}$. Using the base relations $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$ gives us the intended result:*

$$\{j\} \sqsupset_{F_5}^{\mathsf{Conflicts}} \{h,i,j\}$$

However, $\sqsupseteq^{\mathsf{Conflicts}}$ does not fully model argumentative reasoning about this AF, since $\{j\}$ is equally as plausible to be accepted as the admissible set $\{h,j\}$ with respect to $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$. We argue that this should not be the case, as $\{j\}$ is not as plausible to be accepted as any admissible set. Using $\sqsupseteq_{F_5}^{\mathsf{Undef}}$ gives us the intended result of:

$$\{h,j\} \sqsupset_{F_5}^{\mathsf{Undef}} \{j\}$$

But, the sets $\{j\}$ and $\{h,i,j\}$ are incomparable with respect to Undef*. Thus, both $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$ and $\sqsupseteq_{F_5}^{\mathsf{Undef}}$ alone are not sufficient to fully capture argumentative reasoning. Figure 6.2 depicts $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$ and $\sqsupseteq_{F_5}^{\mathsf{Undef}}$ as lattices.*

By combining the two base relations of $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$ and $\sqsupseteq_{F_5}^{\mathsf{Undef}}$, using one base relation after another, we can capture the entire reasoning process for F_5. Figure 6.3 depicts the preorder $\sqsupseteq_{F_5}^{\mathsf{Conflicts},\mathsf{Undef}}$, where $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$ is used first, and if it returns equality, $\sqsupseteq_{F_5}^{\mathsf{Undef}}$ is used next.

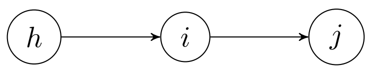

Figure 6.1: Recalled AF F_5 from Example 22.

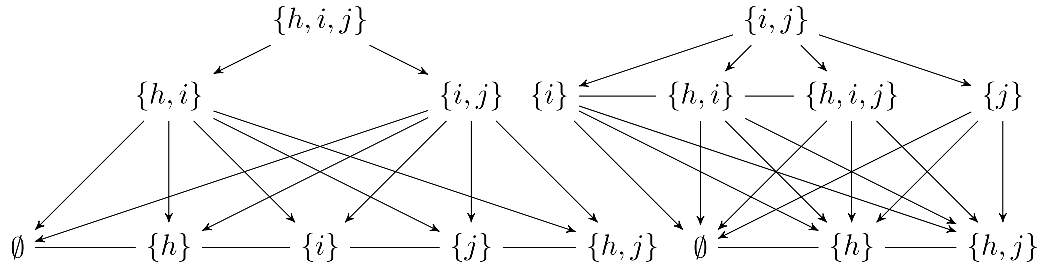

Figure 6.2: Lattices depicting $\sqsupseteq_{F_4}^{\mathsf{Conflicts}}$ (left) and $\sqsupseteq_{F_4}^{\mathsf{Undef}}$ (right) from Example 47, where $E \to E'$ means $E' \sqsupset_{F_5}^{\tau} E$ and $E - E'$ means $E' \equiv_{F_5}^{\tau} E$.

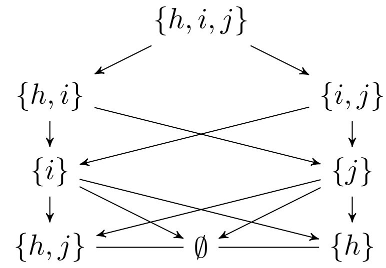

Figure 6.3: $\sqsupseteq_{F_5}^{\mathsf{Conflicts},\mathsf{Undef}}$ from Example 47 depicted as a lattice.

The admissible sets $\emptyset$, $\{h\}$, and $\{h,j\}$ are the most plausible sets with respect to $\sqsupseteq_{F_5}^{\mathsf{Conflicts},\mathsf{Undef}}$. Additionally, we can also state that the set $\{i\}$ is more plausible to be accepted than $\{h,i\}$. Thus, $\sqsupseteq_{F_5}^{\mathsf{Conflicts},\mathsf{Undef}}$ models a ranking of the sets of arguments in F_5 based on their plausibility of acceptance with respect to admissibility.

We propose a general framework for defining extension-ranking semantics using a sequence of base relations.

Definition 41. *Let $F = (A, R)$ be an AF and $(\tau_1, \ldots, \tau_n)$ a sequence of base relations. A* combination method Ω *takes $(\tau_1, \ldots, \tau_n)$ and returns an extension-ranking over $\mathcal{P}(A)$.*

In the rest of this chapter, we discuss two combination methods (*Lexicographic Combination* in Section 6.1 and *Voting Methods* in Section 6.2) and analyse the resulting extension-ranking semantics.

6.1 LEXICOGRAPHIC COMBINATION

In Example 47, we combined $\sqsupseteq^{\mathsf{Conflicts}}$ and $\sqsupseteq^{\mathsf{Undef}}$ by applying these two base relations sequentially. This type of combination is known as *lexicographic combination*.

Definition 42. *Let $F = (A, R)$ be an AF, $E, E' \subseteq A$ and $(\sqsupseteq_F^{\tau_1}, \ldots, \sqsupseteq_F^{\tau_n})$ be sequence of base relations. The* lexicographic combination $\mathsf{lex}(\sqsupseteq_F^{\tau_1}, \ldots, \sqsupseteq_F^{\tau_n})$ *is defined as follows:*

- $E \sqsupset_F^{\mathsf{lex}(\tau_1,\ldots,\tau_n)} E'$ *if and only if there exists* i *such that* $E \sqsupset_F^{\tau_i} E'$ *and for all* $j < i, E \equiv_F^{\tau_j} E'$
- $E \equiv_F^{\mathsf{lex}(\tau_1,\ldots,\tau_n)} E'$ *if and only if for all* $i, E \equiv_F^{\tau_i} E'$.

In other words, we first compare two sets of arguments according to τ_1. If they are equally plausible to be accepted with respect to τ_1, we move on to the next base relation τ_2, continuing until we either find a difference or run out of base relations.

Example 48. *Consider again Example 47. Formally, the lattice in Figure 6.3 is produced by the lexicographic combination of* $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$ *and* $\sqsupseteq_{F_5}^{\mathsf{Undef}}$*, i. e.,* $\mathsf{lex}(\sqsupseteq_{F_5}^{\mathsf{Conflicts}}, \sqsupseteq_{F_5}^{\mathsf{Undef}})$*. For the three sets* $\{h, i\}$*,* $\{i\}$*, and* $\{h, j\}$*, we have:*

$$\{h, j\} \equiv_{F_5}^{\mathsf{Conflicts}} \{i\} \sqsupset_{F_5}^{\mathsf{Conflicts}} \{h, i\}$$

To differentiate between $\{h, j\}$ *and* $\{i\}$*, we use* $\sqsupseteq_{F_5}^{\mathsf{Undef}}$ *next. Therefore we get:*

$$\{h, j\} \sqsupset_{F_5}^{\mathsf{Undef}} \{i\}$$

This entails the final preorder:

$$\{h, j\} \sqsupset_{F_5}^{\mathsf{lex(Conflicts,Undef)}} \{i\} \sqsupset_{F_5}^{\mathsf{lex(Conflicts,Undef)}} \{h, i\}$$

In the following subsections, we will examine sequences of base relations, which correspond to generalisations of the extension semantics discussed by Dung (1995).

6.1.1 Admissible Extension-ranking Semantics

Revisiting the definition of admissibility, a set E is admissible if and only if E has no internal conflicts and contains no undefended argument. These two aspects are captured by Conflicts and Undef, respectively. A combination of these two base relations models a more finer-grained version of admissibility, as we will see later.

Definition 43. *Let* $F = (A, R)$ *be an AF and* $E, E' \subseteq A$*. Define the* admissible extension-ranking semantics r-ad *via:*

$$E \sqsupseteq_F^{\mathsf{r\text{-}ad}} E' \textit{ if and only if } E \sqsupseteq_F^{\mathsf{lex(Conflicts,Undef)}} E'$$

In other words, the set E is more plausible to be accepted than E' if E is more plausible to be accepted than E' with respect to $\sqsupseteq^{\mathsf{Conflicts}}$ or, in the case of equality, E is more plausible to be accepted than E' with respect to $\sqsupseteq^{\mathsf{Undef}}$ in an abstract argumentation framework F. We rename $\mathsf{lex}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ to r-ad to clarify the behaviour of the preorder and align with other works where r-ad was previously defined such as (Skiba et al., 2021b).

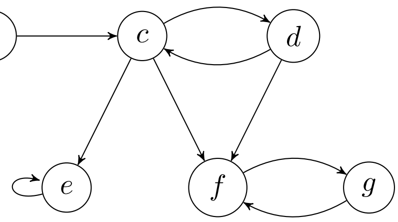

Figure 6.4: Recalled AF F_4 from Example 6.

Example 49. *Consider F_4 from Example 6, depicted in Figure 6.4. When comparing the sets $\{a, b\}$, $\{b\}$, and $\{b, f\}$, we see that $\{b\}$ and $\{b, f\}$ are conflict-free, while $\{a, b\}$ is not. Using $\sqsupset^{\mathsf{Conflicts}}_{F_4}$, we compare these sets using the conflicts base relation. Further comparing $\{b\}$ and $\{b, f\}$, we see that neither of these sets is admissible. Utilising $\sqsupset^{\mathsf{Undef}}_{F_4}$, we get:*

$$\mathsf{Undef}_{F_4}(\{b\}) = \{b\} \quad \textit{and} \quad \mathsf{Undef}_{F_4}(\{b, f\}) = \{b, f\}$$

Thus, we get:

$$\{b\} \sqsupset^{\mathsf{r\text{-}ad}}_{F_4} \{b, f\}$$

This indicates that $\{b\}$ is closer to being admissible than $\{b, f\}$. Extending our investigation to the three sets, we get:

$$\{b\} \sqsupset^{\mathsf{r\text{-}ad}}_{F_4} \{b, f\} \sqsupset^{\mathsf{r\text{-}ad}}_{F_4} \{a, b\}$$

For the admissible sets $\{a, c, g\}$ and $\{a, d, g\}$, we have:

$$\begin{aligned}&\mathsf{Conflicts}_{F_4}(\{a, c, g\}) = \mathsf{Conflicts}_{F_4}(\{a, d, g\}) = \emptyset \quad \textit{and}\\ &\mathsf{Undef}_{F_4}(\{a, c, g\}) = \mathsf{Undef}_{F_4}(\{a, d, g\}) = \emptyset\end{aligned}$$

This implies:

$$\{a, c, g\} \equiv^{\mathsf{r\text{-}ad}}_{F_4} \{a, d, g\}$$

Importantly, $\{a, c, g\}$ and $\{a, d, g\}$ are among the most plausible sets, $\{a, c, g\}, \{a, d, g\} \in \mathtt{max}_{\mathsf{r\text{-}ad}}(F_4)$.

For the two conflict-free sets $\{b, g\}$ and $\{a, f\}$, we get:

$$\mathsf{Undef}_{F_4}(\{b, g\}) = \{b\} \quad \textit{and} \quad \mathsf{Undef}_{F_4}(\{a, f\}) = \{f\}$$

Therefore, these two sets are incomparable:

$$\{b, g\} \asymp^{\mathsf{r\text{-}ad}}_{F_4} \{a, f\}$$

The full preorder of $\sqsupset^{\mathsf{r\text{-}ad}}_{F_5}$ from Example 47 can be found in Figure 6.3. The three admissible sets $\{h, j\}$, $\emptyset$, and $\{h\}$ are the most plausible sets, with every non-admissible set ranked lower. The set $\{i\}$ is closer to being admissible than $\{h, i\}$.

Next, we show the compliance of the admissible extension-ranking semantics with the principles defined in Chapter 4.

Since r-ad is the lexicographic combination of $\sqsupseteq^{\mathsf{Conflicts}}$ and $\sqsupseteq^{\mathsf{Undef}}$, we first check for minimal conflicts and then, in case of equality, for minimal undefended arguments. A set E is admissible if both of these base relations return $\emptyset$ for E.

Proposition 25. r-ad *satisfies* ad-generalisation. *Proof: P. 184*

r-ad has a strict relationship between conflicting and conflict-free sets, such that conflict-free sets are strictly more plausible to be accepted than conflicting sets, implying that *respect conflicts* is satisfied.

Proposition 26. r-ad *satisfies* respect conflicts. *Proof: P. 184*

Lemma 2 and Proposition 16 show that the base relations Conflicts, Undef, Condef, and Unatt satisfy *composition* and *decomposition*. Using this information, we can show that the lexicographic combination of these base relations also satisfies *composition* and *decomposition*.

Proposition 27. r-ad *satisfies* composition *and* decomposition. *Proof: P. 185*

The *Fundamental Lemma* of Dung (1995) already induces reinstatement, meaning that the addition of a defended argument to an admissible set E should not hurt the admissibility of E. Thus, *weak reinstatement* is satisfied, and *strong reinstatement* is violated because ad-*generalisation* is satisfied, which is desirable since we get admissibility and not complete extension semantics.

Proposition 28. r-ad *satisfies weak* weak reinstatement. *Proof: P. 185*

By adding an outgoing attack from an admissible set to an abstract argumentation framework, we do not break the admissibility of any set. Thus, *addition robustness* is satisfied.

Proposition 29. r-ad *satisfies* addition robustness. *Proof: P. 185*

For r-ad, we apply $\sqsupseteq^{\mathsf{Conflicts}}$ and then $\sqsupseteq^{\mathsf{Undef}}$ to generate the final preorder. However, if we reversed the order and apply $\sqsupseteq^{\mathsf{Undef}}$ first and then $\sqsupseteq^{\mathsf{Conflicts}}$, conflicting sets like A would be ranked quite high.

Example 50. *Consider F_4 from Example 6. For any admissible set E, we still get:*

$$\mathsf{Undef}_{F_4}(\{E\}) = \emptyset \quad \textit{and} \quad \mathsf{Conflicts}_{F_4}(\{E\}) = \emptyset$$

So E is among the most plausible sets for lex(Undef, Conflicts).

If we consider the sets $\{a, b, c, d, e, f, g\}$ and $\{b, f\}$, then:

$$\mathsf{Undef}_{F_4}(\{a, b, c, d, e, f, g\}) = \{b\} \quad \textit{and} \quad \mathsf{Undef}_{F_4}(\{b, f\}) = \{b, f\}$$

This implies:

$$\{a, b, c, d, e, f, g\} \sqsupset_{F_4}^{\mathsf{lex(Undef,Conflicts)}} \{b, f\}$$

Although $\{b, f\}$ is conflict-free, $\{a, b, c, d, e, f, g\}$ is not, meaning that respect conflicts *is violated by* lex(Undef, Conflicts).

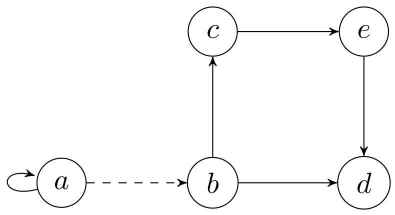

Figure 6.5: Recalled AF F_{12} from Example 41, the attack (a, b) is added later to obtain F'_{12}.

Respect conflicts is not the only principle violated by $\mathsf{lex}(\mathsf{Undef}, \mathsf{Conflicts})$, which is satisfied by $\mathsf{lex}(\mathsf{Conflicts}, \mathsf{Undef})$. The principle *addition robustness* is also violated.

Example 51. *Consider AF F_{12} from Example 41, recalled in Figure 6.5, and the sets $\{a, d\}$ and $\{a, b, c, d\}$. In Example 41, we have seen that:*

$$\{a, b\} \sqsupset_{F_{12}}^{\mathsf{Undef}} \{a, b, c, d\}$$

We add the attack (a, b) to create F'_{12}. However, in this AF, we get:

$$\{a, b, c, d\} \sqsupset_{F'_{12}}^{\mathsf{Undef}} \{a, b\}$$

So, addition robustness *is violated by* $\mathsf{lex}(\mathsf{Undef}, \mathsf{Conflicts})$.

By applying the base relation $\sqsupseteq^{\mathsf{Conflicts}}$ first, we enforce that conflicting sets are less plausible to accepted than conflict-free sets. Therefore, we will focus on sequences of base relations, where $\sqsupseteq^{\mathsf{Conflicts}}$ is applied first.

6.1.2 Complete Extension-ranking Semantics

Complete extensions are admissible sets in which every defended argument is included. A generalisation of the complete extension semantics should be based on the admissible extension-ranking semantics and refined by favouring sets that contain more defended arguments. This behaviour can be modelled with a lexicographic combination of $\sqsupseteq^{\mathsf{r\text{-}ad}}$ and $\sqsupseteq^{\mathsf{Condef}}$.

Definition 44. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. Define the* complete extension-ranking semantics $\mathsf{r\text{-}co}$ *via:*

$$E \sqsupseteq_F^{\mathsf{r\text{-}co}} E' \textit{ if and only if } E \sqsupseteq_F^{\mathsf{lex}(\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef})} E'$$

If E is strictly more plausible to be accepted than E' with respect to $\sqsupseteq^{\mathsf{r\text{-}ad}}$, then this relation also holds for $\mathsf{r\text{-}co}$. If both sets are equally plausible to be accepted with respect to $\sqsupseteq^{\mathsf{r\text{-}ad}}$, then the base relation $\sqsupseteq^{\mathsf{Condef}}$ is used.

Example 52. *Consider F_4 from Example 6 and the sets $\{b\}$, $\{g\}$, and $\{d\}$. While $\{g\}$ and $\{d\}$ are both admissible, the set $\{b\}$ is not admissible. Using $\sqsupseteq^{\mathsf{r\text{-}ad}}$, we get:*

$$\{g\} \sqsupset_{F_4}^{\mathsf{r\text{-}co}} \{b\} \quad \textit{and} \quad \{d\} \sqsupset_{F_4}^{\mathsf{r\text{-}co}} \{b\}$$

$$\begin{tikzcd}
& \{h,i,j\} \arrow[dl] \arrow[dr] & \\
\{h,i\} \arrow[d] \arrow[drr] & & \{i,j\} \arrow[d] \arrow[dll] \\
\{i\} \arrow[dr] & & \{j\} \arrow[dl] \\
& \emptyset \arrow[d] & \\
& \{h\} \arrow[d] & \\
& \{h,j\} &
\end{tikzcd}$$

Figure 6.6: $\sqsupseteq^{\mathsf{r\text{-}co}}_{F_5}$ from Example 53 depicted as a lattice, where $E \to E'$ means $E' \sqsupseteq^{\mathsf{r\text{-}co}}_{F_5} E$.

To further compare $\{g\}$ and $\{d\}$, we examine $\sqsupseteq^{\mathsf{Condef}}$. Argument a is defended by both sets, while $\{d\}$ defends g in addition to a. Thus, we get:

$$\mathsf{Condef}_{F_4}(\{g\}) = \{a\} \quad \textit{and} \quad \mathsf{Condef}_{F_4}(\{d\}) = \{a, g\}$$

Therefore we have:

$$\{g\} \sqsupset^{\mathsf{r\text{-}co}}_{F_4} \{d\}$$

Since $\sqsupseteq^{\mathsf{r\text{-}co}}$ is built on top of $\sqsupseteq^{\mathsf{r\text{-}ad}}$, we can further extend the result of Example 49:

$$\{g\} \sqsupset^{\mathsf{r\text{-}co}}_{F_4} \{d\} \sqsupset^{\mathsf{r\text{-}co}}_{F_4} \{b\} \sqsupset^{\mathsf{r\text{-}co}}_{F_4} \{b, f\} \sqsupset^{\mathsf{r\text{-}co}}_{F_4} \{a, b\}$$

Among these sets, $\{g\}$ is the closed set to be complete, although it is not complete itself.

Example 53. *Consider again F_5 from Example 47. The lattice in Figure 6.6 depicts the preorder $\sqsupseteq^{\mathsf{r\text{-}co}}_{F_5}$. The first three levels are the same as in the preorder $\sqsupseteq^{\mathsf{r\text{-}ad}}_{F_5}$, but with $\sqsupseteq^{\mathsf{r\text{-}co}}_{F_5}$, we can distinguish the sets $\emptyset$, $\{h\}$, and $\{h, j\}$. The complete set $\{h, j\}$ is the most plausible set, and between the two admissible sets $\emptyset$ and $\{h\}$, we can say that $\{h\}$ is closer to being complete than $\emptyset$.*

The complete extension semantics is a refinement of admissibility, so it makes intuitive sense to refine $\sqsupseteq^{\mathsf{r\text{-}ad}}$ with an additional base relation. In this case, we use $\sqsupseteq^{\mathsf{Condef}}$, and this combination indeed serves as a generalisation of the complete extension semantics.

Proposition 30. r-co *satisfies* co-generalisation. *Proof: P. 185*

Since our lexicographic combination starts with $\sqsupseteq^{\mathsf{Conflicts}}$, it is clear that there is a distinction between conflict-free and conflicting sets of arguments, thus satisfying *respect conflicts*.

Proposition 31. r-co *satisfies* respect conflicts. *Proof: P. 186*

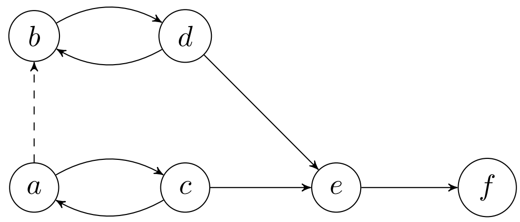

Figure 6.7: AF F_{14} from Example 54, the dashed attack from a to b is added to obtain F'_{14}.

Similar to r-ad, the lexicographic combination of $\sqsupseteq^{\mathsf{Conflicts}}$, $\sqsupseteq^{\mathsf{Undef}}$, and $\sqsupseteq^{\mathsf{Condef}}$ does not violate *composition* and *decomposition*.

Proposition 32. r-co *satisfies* composition *and* decomposition. *Proof: P. 186*

The main motivation for the complete extension semantics is that every defended argument should be accepted, thus a generalisation of the complete extension semantics should satisfy *strong reinstatement*.

Proposition 33. r-co *satisfies* strong reinstatement. *Proof: P. 186*

Adding an attack can lead to additional defended arguments, meaning that complete sets may no longer be complete after adding an attack originating from that set. Thus, the complete extension-ranking semantics violates *addition robustness*.

Example 54. *Let F_{14} be the AF depicted in Figure 6.7. Consider the two sets $\{a\}$ and $\{b, c\}$. $\{a\}$ is a complete extension, while $\{b, c\}$ is not a complete extension, so:*

$$\{a\} \sqsupset^{\mathsf{r\text{-}co}}_{F_{14}} \{b, c\}$$

If addition robustness *were satisfied, we could add an attack between $\{a\}$ and $\{b, c\}$ and maintain this relationship. Suppose we add (a, b) to F_{14} to create F'_{14}, as shown with the dashed line in Figure 6.7. Then $\{a\}$ is no longer a complete extension, since the arguments d and f are defended, implying $\mathsf{Condef}_{F'_{14}}(\{a\}) = \{d, f\}$, while for $\{b, c\}$ we have $\mathsf{Condef}_{F'_{14}}(\{b, c\}) = \{f\}$. Therefore:*

$$\{b, c\} \sqsupset^{\mathsf{r\text{-}co}}_{F'_{14}} \{a\}$$

This violates addition robustness.

6.1.3 Grounded Extension-ranking Semantics

The grounded extension is defined as the minimal complete extension. It is natural to refine the complete extension-ranking semantics by incorporating a notion of minimality to obtain a generalisation of the grounded extension semantics. The *subset relation* $\subseteq$ directly models this notion of minimality and can be used as a base relation. In other words, a set E is more plausible to be accepted than E' if E is smaller

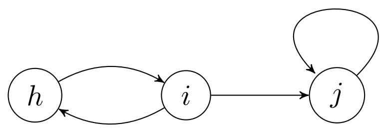

Figure 6.8: AF F_{15} from Example 56.

(with respect to the subset comparisons) than E'. Note that this base relation is only useful in combination with other base relation, as the structure of the underlying abstract argumentation framework is completely irrelevant for determining the plausibility of acceptance of a set with respect to $\subseteq$.

Now, we can define a generalisation of the grounded extension semantics by refining $\sqsupseteq^{\text{r-co}}$ with $\subseteq$.

Definition 45. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. Define* grounded extension-ranking semantics r-gr *via:*

$$E \sqsupseteq_F^{\text{r-gr}} E' \text{ if and only if } E \sqsupseteq_F^{\text{lex(Conflicts,Undef,Condef},\subseteq)} E'$$

Example 55. *Consider F_4 from Example 6 and the sets $\{g\}$, $\{a, g\}$, and $\{a, c, g\}$. $\{a, g\}$ and $\{a, c, g\}$ are both complete extensions, while $\{g\}$ is not complete. Hence, $\sqsupseteq^{\text{r-co}}$ gives us:*

$$\{a, g\} \sqsupset_{F_4}^{\text{r-gr}} \{g\} \quad \text{and} \quad \{a, c, g\} \sqsupset_{F_4}^{\text{r-gr}} \{g\}$$

$\{a, g\}$ and $\{a, c, g\}$ are not the grounded extension, however $\{a, g\} \subset \{a, c, g\}$, which means:

$$\{a, g\} \sqsupset_{F_4}^{\text{r-gr}} \{a, c, g\}$$

Using the results from Example 52, we can extend the ranking:

$$\{a, g\} \sqsupset_{F_4}^{\text{r-gr}} \{a, c, g\} \sqsupset_{F_4}^{\text{r-gr}} \{g\} \sqsupset_{F_4}^{\text{r-gr}} \{d\} \sqsupset_{F_4}^{\text{r-gr}} \{b\} \sqsupset_{F_4}^{\text{r-gr}} \{b, f\} \sqsupset_{F_4}^{\text{r-gr}} \{a, b\}$$

Example 56. *Let $F_{15} = (\{h, i, j\}, \{(h, i), (i, h), (i, j), (j, j)\})$ be an AF, as depicted in Figure 6.8. The full preorders of $\sqsupseteq_{F_{15}}^{\text{r-co}}$ (left) and $\sqsupseteq_{F_{15}}^{\text{r-gr}}$ (right) are depicted as lattices in Figure 6.9. The main difference between these two preorders is that $\emptyset$ ranks better than $\{h\}$ and $\{i\}$ in $\sqsupseteq_{F_{15}}^{\text{r-gr}}$. Although $\{h\}$ and $\{i\}$ are not the grounded extension, these two sets are still considered more plausible to be accepted than every other set except $\emptyset$ because they are complete extensions.*

Note that the grounded extension is by definition a complete extension. So, it should hold that a set of arguments E that is more plausible to be accepted than another set E' with respect to the complete extension-ranking semantics is also more plausible to be accepted than E' with respect to the grounded extension-ranking semantics. These two extension-ranking semantics mirror the behaviour of Dung's complete and grounded extension semantics.

The unique grounded extension is the smallest set among the complete extensions. Creating a preorder based on $\sqsupseteq^{\text{r-co}}$, which is then refined based on the size of the sets, entails a generalisation of the grounded extension semantics.

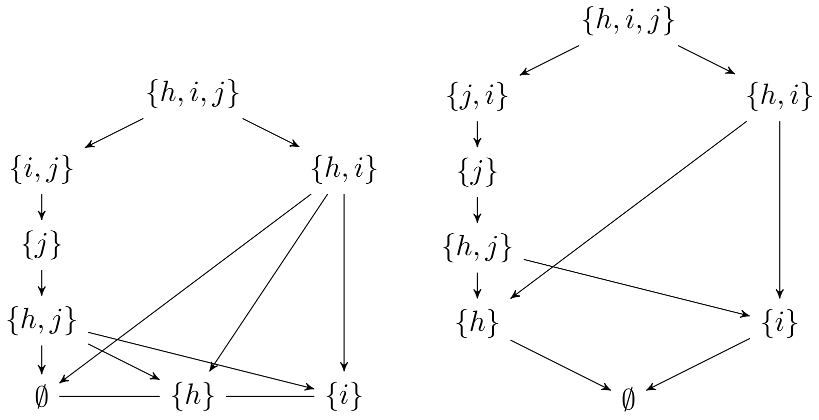

Figure 6.9: Lattices depicting $\sqsupseteq^{\text{r-co}}_{F_{15}}$ (left) and $\sqsupseteq^{\text{r-gr}}_{F_{15}}$ (right) from Example 56.

Proposition 34. r-gr *satisfies* gr-generalisation. *Proof: P. 186*

Since r-gr is based on r-co, we can infer the properties of r-gr via r-co. r-co satisfies *respect conflicts*, *composition*, *decomposition*, and *strong reinstatement*, we know that r-gr satisfies these principles as well, and also violates *addition robustness*.

Proposition 35. r-gr *satisfies* respect conflicts*,* composition*,* decomposition*, and* strong reinstatement*.* r-gr *violates* addition robustness. *Proof: P. 187*

6.1.4 Preferred Extension-ranking Semantics

The preferred extension semantics uses a maximisation approach to reason. More precisely, the preferred extensions are the maximally admissible sets. We can model this using the *superset relation* $\supseteq$, where a superset E is more plausible to be accepted than its subset E'.

To define a generalisation of the preferred extension semantics, we lexicographically combine $\sqsupseteq^{\text{r-ad}}$ with $\supseteq$.

Definition 46. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. Define* preferred extension-ranking semantics r-pr *via:*

$$E \sqsupseteq^{\text{r-pr}}_F E' \text{ if and only if } E \sqsupseteq^{\text{lex(Conflicts,Undef,}\supseteq)}_F E'$$

Example 57. *Continuing with Example 6, consider the sets $\{b\}$, $\{a\}$, and $\{a, g\}$. $\{a\}$ and $\{a, g\}$ are admissible sets, while $\{b\}$ is not admissible. Based on* r-ad*, we have:*

$$\{a\} \sqsupset^{\text{r-pr}}_{F_4} \{b\} \quad \textit{and} \quad \{a, g\} \sqsupset^{\text{r-pr}}_{F_4} \{b\}$$

Additionally, both $\{a\}$ and $\{a, g\}$ are not preferred, but $\{a\} \subset \{a, g\}$, so:

$$\{a, g\} \sqsupset^{\text{r-pr}}_{F_4} \{a\}$$

$$\begin{tikzcd}
& \{h,i,j\} \arrow[dl] \arrow[dr] & \\
\{i,j\} \arrow[d] & & \{h,i\} \arrow[dddl] \\
\{j\} \arrow[d] & & \\
\{h,j\} \arrow[dr] & & \\
& \emptyset \arrow[dl] \arrow[dr] & \\
\{h\} \arrow[rr, dash] & & \{i\}
\end{tikzcd}$$

Figure 6.10: Lattice depicting $\sqsupseteq^{\text{r-pr}}_{F_{15}}$ from Example 56.

This means $\{a, g\}$ is closer to being preferred than $\{a\}$. Extending the ranking from Example 49, we get:

$$\{a, g\} \sqsupset^{\text{r-pr}}_{F_4} \{a\} \sqsupset^{\text{r-pr}}_{F_4} \{b\} \sqsupset^{\text{r-pr}}_{F_4} \{b, f\} \sqsupset^{\text{r-pr}}_{F_4} \{a, b\}$$

Example 58. *Consider F_{15} from Example 56. The preorder $\sqsupseteq^{\text{r-pr}}_{F_{15}}$ is depicted in Figure 6.10. If we compare the lattice for $\sqsupseteq^{\text{r-pr}}_{F_{15}}$ with the lattice for $\sqsupseteq^{\text{r-co}}_{F_{15}}$, we see that the most plausible sets with respect to $\sqsupseteq^{\text{r-co}}_{F_{15}}$ are split. $\emptyset$ is ranked worse than $\{h\}$ and $\{i\}$, which aligns with the preferred extension semantics, as $\emptyset$ is not a preferred extension, while $\{h\}$ and $\{i\}$ are.*

The preferred extensions can also be characterised as maximally complete extensions. For extension semantics, this change in definition does not alter the acceptance of any set, since every preferred extension is also a complete extension (Baroni et al., 2018). However, defining the preferred extension-ranking semantics based on completeness rather than admissibility yields different extension rankings.

Definition 47. *Let $F = (A, R)$ and $E, E' \subseteq A$. We define the* complete-preferred extension-ranking semantics r-co-pr *via:*

$$E \sqsupseteq^{\text{r-co-pr}}_{F} E' \text{ if and only if } E \sqsupseteq^{\text{lex}(\text{Conflicts},\text{Undef},\text{Condef},\supseteq)}_{F} E'$$

Example 59. *Consider F_4 from Example 6 and sets $\{d\}$ and $\emptyset$. Using* r-pr*, we obtain:*

$$\{d\} \sqsupset^{\text{r-pr}}_{F_4} \emptyset$$

since both sets are admissible and $\{d\}$ is the larger set. However, using $\sqsupseteq^{\text{r-co-pr}}$, we get:

$$\emptyset \sqsupset^{\text{r-co-pr}}_{F_4} \{d\}$$

Since these two sets are not complete, but $\mathsf{Condef}_{F_4}(\{d\}) = \{a, g\}$ and $\mathsf{Condef}_{F_{15}}(\emptyset) = \{a\}$. Switching from admissibility to completeness changes our rankings.

As another example, consider the sets $\{a, g\}$, $\{a, c\}$, and $\emptyset$. All three sets are admissible, so they are equally plausible to be accepted with respect to $\sqsupseteq_{F_4}^{\text{r-ad}}$. However, $\emptyset \subset \{a, g\}$ and $\emptyset \subset \{a, c\}$, therefore we have:

$$\{a, g\} \sqsupset_{F_4}^{\text{r-pr}} \emptyset \quad \text{and} \quad \{a, c\} \sqsupset_{F_4}^{\text{r-pr}} \emptyset$$

But $\{a, g\}$ and $\{a, c\}$ are incomparable with respect to $\subseteq$. We get:

$$\{a, g\} \asymp_{F_4}^{\text{r-pr}} \{a, c\} \sqsupset_{F_4}^{\text{r-pr}} \emptyset$$

For $\sqsupseteq_{F_4}^{\text{r-co-pr}}$, we compare $\{a, g\}$, $\{a, c\}$, and $\emptyset$ with $\sqsupseteq^{\text{Condef}}$. $\{a, g\}$ is a complete extension, so $\text{Condef}_{F_4}(\{a, g\}) = \emptyset$. Since $\text{Condef}_{F_4}(\{a, c\}) = g$ and $\text{Condef}_{F_4}(\emptyset) = a$, the other two sets are not complete. Among these three sets, $\{a, g\}$ is the closest one to be preferred, while the other two are incomparable.

$$\{a, g\} \sqsupset_{F_4}^{\text{r-co-pr}} \{a, c\} \asymp_{F_4}^{\text{r-co-pr}} \emptyset$$

Example 59 shows that r-pr and r-co-pr are not refinements of each other and produce different rankings.

Despite the differences between r-pr and r-co-pr, both satisfy pr-generalisation. Therefore, depending on the context and application, either version can be used as a generalisation of the preferred extension semantics. When deciding which version to use, the importance of completeness of sets must be considered. If sets that are consistently defended and contain more arguments are to be favoured, then r-co-pr should be used; otherwise r-pr can be used instead.

Proposition 36. r-pr *and* r-co-pr *satisfy* pr-generalisation. *Proof: P. 187*

Since Conflicts is the first base relation used for both r-pr and r-co-pr, this implies that *respect conflicts* is satisfied.

Proposition 37. r-pr *and* r-co-pr *satisfy* respect conflicts. *Proof: P. 187*

Similar to the other extension-ranking semantics, a lexicographic combination of $\sqsupseteq_F^{\text{r-ad}}$ or $\sqsupseteq_F^{\text{r-co}}$ and maximisation satisfies *composition* and *decomposition*.

Proposition 38. r-pr *and* r-co-pr *satisfy* composition *and* decomposition. *Proof: P. 187*

The main difference between $\sqsupseteq_F^{\text{r-ad}}$ and $\sqsupseteq_F^{\text{r-pr}}$ (or $\sqsupseteq_F^{\text{r-co-pr}}$) is maximisation in the preferred case. Adding an argument to a set that is defended and does not create new conflicts should make the set more plausible to be accepted based on preferred reasoning, implying that the reinstatement principles are satisfied.

Proposition 39. r-pr *and* r-co-pr *satisfy* strong reinstatement. *Proof: P. 187*

To show that *addition robustness* is violated by r-pr and r-co-pr, we use Example 31 again.

6.1.5 (Semi)-Stable Extension-ranking Semantics

For a set of arguments to be stable, it must be conflict-free and attack all arguments not contained in that set. We have already shown in Proposition 4 that there is no extension-ranking semantics that satisfies st-soundness, and therefore st-generalisation is impossible to satisfy. Although we could consider defining an extension-ranking semantics by lexicographically combining $\sqsupseteq^{\mathsf{Conflicts}}$ and $\sqsupseteq^{\mathsf{Unatt}}$, the resulting semantics will not be st-sound. However, a generalisation of the semi-stable extension semantics is possible. Recall that a set E is semi-stable if it is complete and maximises $E \cup E^{+}$. We achieve a generalisation by lexicographically combining $\sqsupseteq^{\mathsf{r\text{-}co}}$ and $\sqsupseteq^{\mathsf{Unatt}}$.

Definition 48. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. We define the* semi-stable extension-ranking semantics r-sst *via:*

$$E \sqsupseteq^{\mathsf{r\text{-}sst}}_{F} E' \text{ if and only if } E \sqsupseteq^{\mathsf{lex(Conflicts,Undef,Condef,Unatt)}}_{F} E'$$

Example 60. *Consider F_4 from Example 6 and the sets $\{a, g\}$, $\{a, d, g\}$, and $\{g\}$. $\{a, g\}$ and $\{a, d, g\}$ are both complete extensions, while $\{g\}$ is not complete. With $\sqsupseteq^{\mathsf{r\text{-}co}}_{F_4}$, we get:*

$$\{a, g\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{g\} \quad \text{and} \quad \{a, d, g\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{g\}$$

To further compare $\{a, g\}$ and $\{a, d, g\}$, we use $\sqsupseteq^{\mathsf{Unatt}}_{F_4}$. Both sets are not semi-stable, but:

$$\mathsf{Unatt}_{F_4}(\{a, g\}) = \{c, d, e\} \quad \text{and} \quad \mathsf{Unatt}_{F_4}(\{a, d, g\}) = \{e\}$$

Thus we have:

$$\{a, d, g\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{a, g\}$$

If we compare the semi-stable extension $\{a, c, g\}$ with $\{a, d, g\}$, we get:

$$\{a, c, g\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{a, d, g\}$$

because $\mathsf{Unatt}_{F_4}(\{a, c, g\}) = \emptyset$, implying $\{a, c, g\} \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F_4)$. Since $\sqsupseteq^{\mathsf{r\text{-}sst}}_{F_4}$ is based on $\sqsupseteq^{\mathsf{r\text{-}co}}_{F_4}$, we can extend the ranking computed in Example 52:

$$\begin{aligned}\{a, c, g\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{a, d, g\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{a, g\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{g\} \\ \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{d\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{b\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{b, f\} \sqsupset^{\mathsf{r\text{-}sst}}_{F_4} \{a, b\}\end{aligned}$$

Example 60 illustrates the behaviour of this family of extension-ranking semantics nicely. We can divide the resulting ranking into parts: first, the (semi-) stable extension $\{a, c, g\}$; then, complete extensions that are not semi-stable $\{a, d, g\}$ and $\{a, g\}$; followed by admissible sets $\{g\}$ and $\{d\}$; then conflict-free sets $\{b\}$ and $\{b, f\}$; and ending with the conflicting set $\{a, b\}$.

Example 61. *Consider F_{15} from Example 56. The corresponding lattice to $\sqsupseteq^{\mathsf{r\text{-}sst}}_{F_{15}}$ is depicted in Figure 6.11. Comparing $\sqsupseteq^{\mathsf{r\text{-}sst}}_{F_{15}}$ with $\sqsupseteq^{\mathsf{r\text{-}co}}_{F_{15}}$, we see that the most plausible sets of $\sqsupseteq^{\mathsf{r\text{-}co}}_{F_{15}}$ are split. $\{i\}$ is more plausible to be accepted than $\{h\}$, and $\{h\}$ is more plausible to be accepted than $\emptyset$. Both $\{h\}$ and $\emptyset$ are complete extensions, but neither is a semi-stable extension. However, we show that $\{h\}$ is closer to being semi-stable than $\emptyset$.*

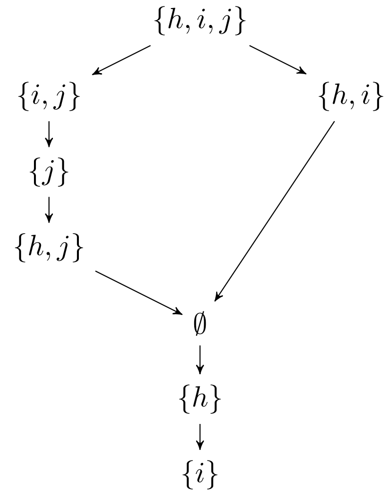

Figure 6.11: Lattice depicting $\sqsupseteq^{\mathsf{r\text{-}sst}}_{F_{15}}$ from Example 61.

An important property of the semi-stable extensions is that these extensions coincide with the stable extensions as long as stable extensions exist. A similar behaviour can be observed for the semi-stable extension-ranking semantics: if stable extensions exist, then the most plausible sets of the semi-stable extension-ranking semantics are the stable extensions.

Proposition 40. *Let $F = (A, R)$ be an AF with $\mathsf{st}(F) \neq \emptyset$, then $\mathtt{max}_{\mathsf{r\text{-}sst}}(F) = \mathsf{st}(F)$.*

Proof: P. 188

Proposition 40 also implies that r-sst satisfies st-*completeness*, even if the abstract argumentation framework has no stable extension.

Proposition 41. r-sst *satisfies* st-completeness. *Proof: P. 188*

Note that $\mathsf{Unatt}_F(E) \neq \emptyset$ for AF $F = (A, R)$ and $E \subseteq A$ does not necessarily mean that E is not among the most plausible sets.

Example 62. *Consider AF $F_{16} = (\{a, b, c, d\}, \{(a, b), (b, c), (d, d)\})$ as depicted in Figure 6.12. Consider the set $\{a, c\}$. This set is a complete extension, i. e.,*

$$\mathsf{Conflicts}_{F_{16}}(\{a, c\}) = \mathsf{Undef}_{F_{16}}(\{a, c\}) = \mathsf{Condef}_{F_{16}}(\{a, c\}) = \emptyset$$

Argument d is not attacked by $\{a, c\}$, so $\mathsf{Unatt}_{F_{16}}(\{a, c\}) = \{d\}$. However, the only argument attacking d is d itself. Any set containing d is not conflict-free, therefore there cannot be a set E with $\mathsf{Conflicts}_{F_{16}}(E) = \emptyset$ and $\mathsf{Unatt}_{F_{16}}(E) = \emptyset$. Showing that $\{a, c\} \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F_{16})$.

Refining $\sqsupseteq^{\mathsf{r\text{-}co}}$ with $\sqsupseteq^{\mathsf{Unatt}}$ maximises the range of each set and is therefore a generalisation of the semi-stable extension semantics.

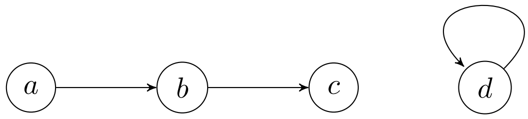

Figure 6.12: AF F_{16} from Example 62.

Principles	r-ad	r-co	r-gr	r-pr	r-co-pr	r-sst
σ-generalisation	✓(σ = ad)	✓(σ = co)	✓(σ = gr)	✓(σ = pr)	✓(σ = pr)	✓(σ = sst)
respect conflicts	✓	✓	✓	✓	✓	✓
composition	✓	✓	✓	✓	✓	✓
decomposition	✓	✓	✓	✓	✓	✓
weak reinstatement	✓	✓	✓	✓	✓	✓
strong reinstatement	✗	✓	✓	✓	✓	✓
addition robustness	✓	✗	✗	✗	✗	✗
syntax independence	✓	✓	✓	✓	✓	✓

Table 6.1: Principles satisfied and violated by r-σ for $\sigma \in \{\text{ad}, \text{co}, \text{gr}, \text{pr}, \text{co-pr}, \text{sst}\}$.

Proposition 42. r-sst *satisfies* sst-generalisation. *Proof: P. 188*

Since r-sst is a refinement of r-co, we can base the proof of the next proposition on the proof for r-co.

Proposition 43. r-sst *satisfies* respect conflicts*,* composition*,* decomposition*, and* strong reinstatement*.* r-sst *violates* addition robustness*.* *Proof: P. 188*

It is clear by definition, that all extension-ranking semantics satisfy *syntax independence*.

Table 6.1 summarises the results of this section. For $\sigma \in \{\text{ad}, \text{co}, \text{pr}, \text{sst}\}$, the proposed extension-ranking semantics satisfies the maximal number of principles possible.

6.1.6 Cardinality-based Instances

We can utilise the cardinality-based variations of the base relations from Section 5.2 to define generalisations of extension semantics, similar to the extension-ranking semantics in the preceding sections. Instead of detailing each sequence of base relations, we will provide a summary.

Definition 49. *Let $F = (A, R)$ and $E, E' \subseteq A$. The* cardinality-based extension-ranking semantics r-c-σ *for $\sigma \in \{\text{ad}, \text{co}, \text{gr}, \text{pr}, \text{co-pr}, \text{sst}\}$ are defined as follows:*

- $E \sqsupseteq_F^{\text{r-c-ad}} E'$ *if and only if* $E \sqsupseteq_F^{\text{lex(c-Conflicts,c-Undef)}} E'$.
- $E \sqsupseteq_F^{\text{r-c-co}} E'$ *if and only if* $E \sqsupseteq_F^{\text{lex(c-Conflicts,c-Undef,c-Condef)}} E'$.
- $E \sqsupseteq_F^{\text{r-c-gr}} E'$ *if and only if* $E \sqsupseteq_F^{\text{lex(c-Conflicts,c-Undef,c-Condef,}\subseteq)} E'$.

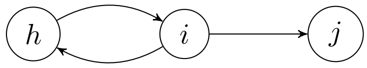

Figure 6.13: AF F_{17} from Example 64.

- $E \sqsupseteq_F^{\textsf{r-c-pr}} E'$ *if and only if* $E \sqsupseteq_F^{\textsf{lex(c-Conflicts,c-Undef,}\supseteq\textsf{)}} E'$.
- $E \sqsupseteq_F^{\textsf{r-c-co-pr}} E'$ *if and only if* $E \sqsupseteq_F^{\textsf{lex(c-Conflicts,c-Undef,c-Condef,}\supseteq\textsf{)}} E'$.
- $E \sqsupseteq_F^{\textsf{r-c-sst}} E'$ *if and only if* $E \sqsupseteq_F^{\textsf{lex(c-Conflicts,c-Undef,c-Condef,c-Unatt)}} E'$.

For each base relation, the cardinality-based variant is used to generalise the corresponding extension semantics similarly to the subset comparison variant. For example, to generalise admissibility, we lexicographically combine $\sqsupseteq^{\textsf{c-Conflicts}}$ and $\sqsupseteq^{\textsf{c-Undef}}$.

Example 63. *Consider F_4 from Example 6 and the sets $\{c, g\}$ and $\{b, f\}$. Argument g is defended by $\{c, g\}$, while c is not defended, and both b, f are not defended by $\{b, f\}$. $\sqsupseteq_{F_4}^{\textsf{r-ad}}$ returns incompatibility between these two sets, since $\{c\}$ is not a subset of $\{b, f\}$. However, using the cardinality-based variant of* r-ad*, we can compare these two sets and say that:*

$$\{c, g\} \sqsupset_{F_4}^{\textsf{r-c-ad}} \{b, f\}$$

Consider the sets $\{a, b\}$, $\{b\}$ and $\{b, f\}$. The set $\{a, b\}$ is not conflict-free, so:

$$\{b\} \sqsupset_{F_4}^{\textsf{r-c-ad}} \{a, b\} \quad \textit{and} \quad \{b, f\} \sqsupset_{F_4}^{\textsf{r-c-ad}} \{a, b\}$$

To further compare $\{b\}$ and $\{b, f\}$, we examine $\sqsupseteq_{F_4}^{\textsf{c-Undef}}$. Argument b is not defended by $\{b\}$, meaning $|\textsf{Undef}_{F_4}(\{b\})| = |\{b\}| = 1$. For $\{b, f\}$, we have $|\textsf{Undef}_{F_4}(\{b, f\})| = |\{b, f\}| = 2$, which gives us:

$$\{b\} \sqsupset_{F_4}^{\textsf{r-c-ad}} \{b, f\}$$

The resulting extension ranking between these three sets coincides with the one for $\sqsupseteq_{F_4}^{\textsf{r-ad}}$ as shown in Example 49.

$$\{b\} \sqsupset_{F_4}^{\textsf{r-c-ad}} \{b, f\} \sqsupset_{F_4}^{\textsf{r-c-ad}} \{a, b\}$$

Example 64. *Consider $F_{17} = (\{h, i, j\}, \{(h, i), (i, h), (i, j)\})$ as depicted in Figure 6.13. The extension rankings for $\sqsupseteq_{F_{17}}^{\textsf{r-ad}}$ and $\sqsupseteq_{F_{17}}^{\textsf{r-c-ad}}$ are depicted in Figure 6.14. The key difference between the two rankings is that $\{h, i\}$ and $\{i, j\}$ are incomparable for $\sqsupseteq_{F_{17}}^{\textsf{r-ad}}$, whereas for $\sqsupseteq_{F_{17}}^{\textsf{r-c-ad}}$, these two sets are comparable, and $\{i, j\}$ is more plausible to be accepted than $\{h, i\}$.*

In Example 63, we observed that $\sqsupseteq_{F_{17}}^{\textsf{r-ad}}$ and $\sqsupseteq_{F_{17}}^{\textsf{r-c-ad}}$ coincide in the upper two levels. We formalise this observation such that $\sqsupseteq_{F_{17}}^{\textsf{r-ad}}$ and $\sqsupseteq_{F_{17}}^{\textsf{r-c-ad}}$ coincide whenever $\sqsupseteq^{\textsf{r-}\sigma}$ can compare two sets.

Proposition 44. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. If $E \sqsupset_F^{\textsf{r-}\sigma} E'$, then $E \sqsupset_F^{\textsf{r-c-}\sigma} E'$ for $\sigma \in \{\textsf{ad}, \textsf{co}, \textsf{gr}, \textsf{pr}, \textsf{co-pr}, \textsf{sst}\}$.* *Proof: P. 189*

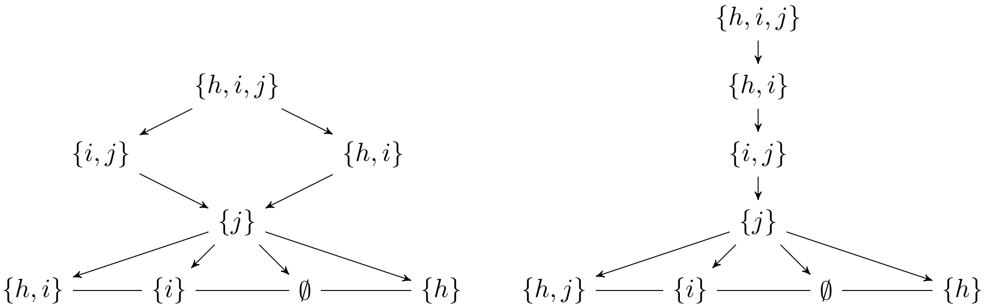

Figure 6.14: Lattices depicting $\sqsupseteq^{\textsf{r-ad}}_{F_{17}}$ (left) and $\sqsupseteq^{\textsf{r-c-ad}}_{F_{17}}$ (right) from Example 64.

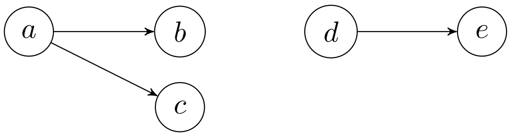

Figure 6.15: AF F_{18} from Example 65.

In other words, the family of extension-ranking semantics r-c-σ is a refinement of r-σ, as it resolves the non-totality of $\sqsupseteq^{\textsf{r-}\sigma}$ for $\sigma \in \{\textsf{ad}, \textsf{co}, \textsf{sst}\}$, but coincides with $\sqsupseteq^{\textsf{r-}\sigma}$ when this extension-ranking semantics can compare two sets.

Next, we examine the principles that the r-c-σ extension-ranking semantics satisfy. Starting with the *σ-generalisation* principle, we see that the cardinality-based variants of the extension-ranking semantics r-c-σ are also generalisations of the extension semantics.

Proposition 45. r-c-σ *satisfies* σ-generalisation $\sigma \in \{\textsf{ad}, \textsf{co}, \textsf{gr}, \textsf{pr}, \textsf{co-pr}, \textsf{sst}\}$.

Proof: P. 189

All the extension-ranking semantics discussed in this subsection use $\sqsupseteq^{\textsf{Conflicts}}$ as their first base relation, thus we see that *respect conflicts* is satisfied.

Proposition 46. r-c-σ *satisfies* respect conflicts *for* $\sigma \in \{\textsf{ad}, \textsf{co}, \textsf{gr}, \textsf{pr}, \textsf{co-pr}, \textsf{sst}\}$.

Proof: P. 189

Next, we see that r-c-σ satisfies *composition*.

Proposition 47. r-c-σ *satisfies* composition *for* $\sigma \in \{\textsf{ad}, \textsf{co}, \textsf{gr}, \textsf{pr}, \textsf{co-pr}, \textsf{sst}\}$.

Proof: P. 189

Unlike r-σ, r-c-σ violates *decomposition* for $\sigma \in \{\textsf{ad}, \textsf{co}, \textsf{gr}, \textsf{pr}, \textsf{co-pr}, \textsf{sst}\}$.

Example 65. *Consider* $F_{18} = (\{a, b, c, d, e\}, \{(a, b), (a, c), (d, e)\})$ *as depicted in Figure 6.15, with* $\{d, e\}$ *and* $\{a, b, c\}$*. This AF can be split into two disjoint subAFs:* $F_{18,1} = (\{a, b, c\}, \{(a, b), (a, c)\})$ *and* $F_{18,2} = (\{d, e\}, \{(d, e)\})$*. Since* $\{a, b, c\}$ *has two conflicts and* $\{d, e\}$ *has only one, it is clear that:*

$$\{d, e\} \sqsupset^{\textsf{c-Conflicts}}_{F_{18}} \{a, b, c\}$$

Given that every r-c-σ *is based on* $\sqsupseteq^{\text{c-Conflicts}}$*, the same holds for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

Examining the subAFs $F_{18,1}$ *and* $F_{18,2}$*, we see that* $\{d, e\} \cap \{a, b, c\}$ *is conflict-free in* $F_{18,1}$*, while* $\{a, b\} \cap \{a, b, c\}$ *has two conflicts, so:*

$$\{d, e\} \cap \{a, b, c\} \sqsupset^{\text{c-Conflicts}}_{F_{18,1}} \{d, e\} \cap \{a, b, c\}$$

However, $E' \cap \{d, e\}$ *is conflict-free in* $F_{18,2}$ *and* $E \cap \{d, e\}$ *has one conflict in* $F_{18,2}$*, so:*

$$\{a, b, c\} \cap \{d, e\} \sqsupset^{\text{c-Conflicts}}_{F_{18,2}} \{d, e\} \cap \{d, e\}$$

This shows that c-Conflicts *violates* decomposition*, thus* r-c-σ *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ *also violates* decomposition*.*

While the cardinality-based extension-ranking semantics are total preorders, they do not satisfy *decomposition*.

Based on our results for r-ad, we show that the cardinality-based variant of the admissibility based extension-ranking semantics only satisfies *weak reinstatement*, but not *strong reinstatement*.

Proposition 48. r-c-ad *satisfies* weak reinstatement *and violates* strong reinstatement*.*
Proof: P. 189

For the remaining semantics $(\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst})$, we show that the cardinality-based extension-ranking semantics satisfies *strong reinstatement*.

Proposition 49. r-c-σ *satisfies* strong reinstatement *for* $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$*.*
Proof: P. 190

Additional robustness is satisfied by r-ad and violated for all other extension-ranking semantics previously discussed using lex as the combination method. Similar results hold for the cardinality-based version r-c-ad.

Proposition 50. r-c-ad *satisfies* addition robustness*.* *Proof: P. 190*

For $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$, the counterexamples used before (Examples 31 and 54) can be reused to show that r-c-σ also violates *addition robustness*. For instance, in Example 54, it holds that $\mathsf{Condef}_{F_{14}}(\{a\}) = \emptyset$ and $\mathsf{Condef}_{F_{14}}(\{b, c\}) = \{d\}$. After adding an attack from $\{a\}$ to $\{b, c\}$, the consistently defended arguments of $\{a\}$ changed to $\mathsf{Condef}_{F'_{14}}(\{a\}) = \{d, f\}$, while the consistently defended arguments of $\{b, c\}$ remained the same, i. e., $\mathsf{Condef}_{F'_{14}}(\{b, d\}) = \{d\}$. Thus, the relation between $\{a\}$ and $\{b, c\}$ in F'_{14} changes to:

$$\{b, d\} \sqsupset^{\text{r-c-}\sigma}_{F'_{14}} \{a\}$$

Therefore, r-c-σ violates *addition robustness* for $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$.

By definition, r-c-σ satisfies *syntax independence* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$, as the names of the individual arguments are irrelevant for calculating the rankings.

Table 6.2 is a summery of this subsection. The cardinality-based extension-ranking semantics behave similarly to their subset variants, but they trade *decomposition* for totality. Depending on the task at hand, we can choose the appropriate variant by evaluating between totality and *decomposition*.

Principles	r-c-ad	r-c-co	r-c-gr	r-c-pr	r-c-co-pr	r-c-sst
σ-generalisation	✓($\sigma = $ ad)	✓($\sigma = $ co)	✓($\sigma = $ gr)	✓($\sigma = $ pr)	✓($\sigma = $ pr)	✓($\sigma = $ sst)
respect conflicts	✓	✓	✓	✓	✓	✓
composition	✓	✓	✓	✓	✓	✓
decomposition	✗	✗	✗	✗	✗	✗
weak reinstatement	✓	✓	✓	✓	✓	✓
strong reinstatement	✗	✓	✓	✓	✓	✓
addition robustness	✓	✗	✗	✗	✗	✗
syntax independence	✓	✓	✓	✓	✓	✓

Table 6.2: Principles satisfied and violated by r-c-σ for $\sigma \in \{\text{ad}, \text{co}, \text{gr}, \text{pr}, \text{co-pr}, \text{sst}\}$.

6.2 VOTING METHODS

Voting theory is an area of research that explores methods and processes for collective decision-making within a group of individuals. A common application are elections, where voters collectively choose the best alternative from a set of alternatives. This selection process aims to be fair by reflecting the voters' preferences. Besides analysing voting systems, voting theory also examines paradoxes and limitations, such as the *Condorcet paradox*, which demonstrates that majority votes can be inconsistent. We begin by recalling essential background information on voting theory and then discuss how it can be applied to rank sets of arguments.

6.2.1 Voting Theory

We start with a formal definition of an *election*.

Definition 50. *An* election $(\mathcal{A}, \mathcal{N})$ *consists of a set* alternatives $\mathcal{A} = \{a_1, \ldots, a_m\}$*, and a set of voters* $\mathcal{N} = \{v_1, \ldots, v_n\}$ *.*

To collectively decide on the alternatives in $\mathcal{A}$, each voter i casts a *vote*.

Definition 51. *In an election* $(\mathcal{A}, \mathcal{N})$*, a voter* $v_i \in \mathcal{N}$ *cast a* vote *as a linear ordering* $\succsim_i$ *of* $\mathcal{A}$*.*

The linear order $\succsim_i$ represents the voter's preference over the alternatives, where $a \succsim_i b$ means voter i prefers or votes for alternative a over alternative b.

A *voting rule* determines the winner(s) of an election, the most preferred alternative based on the votes. Note that we are considering a special case of voting rules that produce a preoder rather than a single best candidate.

Definition 52. *Let* $(\mathcal{A}, \mathcal{N})$ *be an election and* $(\succsim_1, \ldots, \succsim_n)$ *the votes of* $\mathcal{N}$*. A* voting rule $\mathcal{VR}$ *returns a preorder* $\geq^{\mathcal{VR}}$ *over the alternatives* $\mathcal{A}$ *with respect to* $(\succsim_1, \ldots, \succsim_n)$*.*

A number of voting rules can be found in the literature, such as the *plurality voting rule*, where the alternative with the most top votes wins, or the *Borda rule*, where alternatives receive points according to their position in each vote, and the one with the highest total points wins the election. For more information on voting theory and other voting rules, we refer interested readers to (Brandt et al., 2016b).

Next, we introduce the *Copeland rule*, which considers the number of pairwise comparisons that an alternative wins versus the number of pairwise comparisons that the alternative loses. The alternatives with the best win/loss record are the overall winners.

Definition 53. *Let* $(\mathcal{A}, \mathcal{N})$ *be an election and* $(\succsim_1, \ldots, \succsim_n)$ *the votes of* $\mathcal{N}$*. The* Copeland Score $Copeland(x)$ *for alternative* $x \in \mathcal{A}$ *is defined as:*

$$Copeland(x) = \sum_{i=1}^{n} |\{y \in \mathcal{A} \mid x \succsim_i y\}| - |\{y \in \mathcal{A} \mid y \succsim_i x\}|$$

The Copeland Rule $Copeland$ *for two alternatives* $x, y \in \mathcal{A}$ *is then:*

$$x \geq^{Copeland} y \text{ if and only if } Copeland(x) \geq Copeland(y)$$

Example 66. *Consider a small mayoral election with four candidates:* $Ali, Ben, Claus,$ *and* $Dennis$*. Four voters participating* (v_1, v_2, v_3, v_4)*, casting the votes as follows:*

$$\begin{aligned}
v_1 &: Ali \succ_1 Ben \succ_1 Claus \succ_1 Dennis \\
v_2 &: Claus \succ_2 Ali \succ_2 Ben \succ_2 Dennis \\
v_3 &: Dennis \succ_3 Claus \succ_3 Ben \succ_1 Ali \\
v_4 &: Claus \succ_4 Ben \succ_4 Dennis \succ_1 Ali
\end{aligned}$$

Ali *is ranked above* $Claus$ *by* v_1*, but below by the other three voters.* Ali *is ranked above* Ben *and* $Dennis$ *by* v_1 *and* v_2 *but below by* v_3 *and* v_4*, resulting in a Copeland Score of* $Copeland(Ali) = -2$*. The Copeland Scores for the other candidates are:*

$$Copeland(Ben) = 0 \quad Copeland(Claus) = 6 \quad Copeland(Dennis) = -4$$

The final ranking with respect to the Copeland Rule is:

$$Claus \geq^{Copeland} Ben \geq^{Copeland} Ali \geq^{Copeland} Dennis$$

Thus, $Claus$ *is the winner of the election.*

6.2.2 Copeland-based Combination

A voting rule takes a number of votes, presented as linear orders, and returns a preorder over the alternatives. The input-output structure is analogous to extension-ranking semantics, where a sequence of base relations is used to construct a preorder over the powerset of arguments. In this context, the sequence of base relations serves as the votes, and the powerset of arguments represent the alternatives in an election. Thus, a voting rule can be seen as an combination method, forming the basis for a general family of extension-ranking semantics.

Definition 54. *Let $F = (A, R)$ be an AF, $E, E' \subseteq A$ and $(\sqsupseteq_F^{\tau_1}, \ldots, \sqsupseteq_F^{\tau_n})$ being a sequence of base relations. Consider $(\mathcal{P}(A), \{\tau_1, \ldots, \tau_n\})$ as an election and $\mathcal{VR}$ as a voting rule. We define the $\mathcal{VR}$-*based combination *$\mathcal{VR}(\sqsupseteq_F^{\tau_1}, \ldots, \sqsupseteq_F^{\tau_n})$ as follows:*

$$E \sqsupseteq_F^{\mathcal{VR}(\tau_1,\ldots,\tau_n)} E' \text{ if and only if } E \geq^{\mathcal{VR}} E'$$

Note that our votes are binary relations rather than linear orders. Therefore, for voting rules requiring totality and antisymmetry, we must define how to handle these cases separately.

This work focuses on the *Copeland rule*, because of its simplicity and its ability to handle multiple candidates very well. In addition, the *Copeland rule* represents a refinement of the models proposed by Konieczny et al. (2015) to compare σ-extensions using *pairwise comparison criteria*, where a σ-extension E is favoured over E' if E has a better win/loss record than E' with respect to these criteria. Thus, we introduce the *Copeland*-based combination as an instance of the $\mathcal{VR}$-based combination.

Definition 55. *Let $F = (A, R)$ be an AF, $E, E' \subseteq A$ and $(\sqsupseteq_F^{\tau_1}, \ldots, \sqsupseteq_F^{\tau_n})$ be a sequence of base relations. We define the* Copeland-based combination $\mathsf{cope}(\sqsupseteq_F^{\tau_1}, \ldots, \sqsupseteq_F^{\tau_n})$ *via:*

$$E \sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E' \text{ if and only if } E \geq^{Copeland} E'$$

In other words, a set E is more plausible to be accepted than E' if the number of sets ranked worse than E is greater than those ranked worse than E'. This means E has a better balance of wins and losses than E' with respect to the selected base relations.

Example 67. *Consider F_5 from Example 22 and the sets $\{h, i\}$ and $\{i\}$. $\{h, i\}$ wins only against $\{h, i, j\}$ with respect to $\sqsupseteq_{F_5}^{\mathsf{Conflicts}}$ and loses against five other sets ($\{h\}, \{i\}, \{j\}$, $\{h, j\}, \emptyset$), resulting in a value of -4, while $\{i\}$ receives a value of 3. Thus:*

$$\{i\} \sqsupset_{F_5}^{\mathsf{cope(Conflicts)}} \{h, i\}$$

For $\sqsupseteq_{F_5}^{\mathsf{Undef}}$, both $\{h, i\}$ and $\{i\}$ receive a value of -2, leading to:

$$\{h, i\} \equiv_{F_5}^{\mathsf{cope(Undef)}} \{i\}$$

The base relations individually rank these two sets differently. Combining these scores, we get -6 for $\{h, i\}$ and 1 for $\{i\}$, resulting in:

$$\{i\} \sqsupset_{F_5}^{\mathsf{cope(Conflicts,Undef)}} \{h, i\}$$

The full extension ranking of $\sqsupseteq_{F_5}^{\mathsf{cope(Conflicts,Undef)}}$ is depicted as a lattice in Figure 6.16 on the right. Comparing this extension ranking with $\sqsupseteq_{F_5}^{\mathsf{r\text{-}ad}}$. We observe, the set containing every argument $\{h, i, j\}$ is more plausible to be accepted than $\{i, j\}$ with respect to $\sqsupseteq_{F_5}^{\mathsf{cope(Conflicts,Undef)}}$, whereas $\sqsupseteq_{F_5}^{\mathsf{r\text{-}ad}}$, $\{h, i, j\}$ is the least plausible set.

The two different combination methods induce different extension rankings.

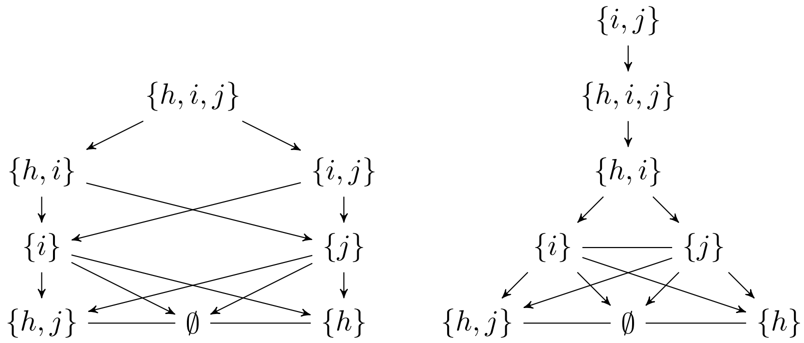

Figure 6.16: Lattices depicting $\sqsupseteq^{\mathsf{r\text{-}ad}}_{F_5}$ (left) and $\sqsupseteq^{\mathsf{cope(Conflicts,Undef)}}_{F_5}$ (right) from Example 67.

The extension-ranking semantics $\sqsupseteq^{\mathsf{cope(nonatt)}}$ and $\sqsupseteq^{\mathsf{cope(strdef)}}$ are refinements of the *Copeland-based extensions* $\mathsf{CBE}_{\sigma,\gamma}$ defined by Konieczny et al. (2015).

Definition 56 (Konieczny et al. (2015)). *Let $F = (A, R)$ be an AF, σ an extension semantics, and $\gamma \in \{\mathsf{nonatt}, \mathsf{strdef}\}$. The set of* Copeland-based extensions $\mathsf{CBE}_{\sigma,\gamma}$ *is defined as:*

$$\mathsf{CBE}_{\sigma,\gamma} = argmax_{E \in \sigma(F)} |\{E' \in \sigma(F) \mid E \sqsupseteq^{\gamma}_{F} E'\}| - |\{E' \in \sigma(F) \mid E' \sqsupseteq^{\gamma}_{F} E\}|$$

Copeland-based extensions are the σ-extensions that rank highest among all σ-extensions with respect to $\gamma \in \{\mathsf{nonatt}, \mathsf{strdef}\}$. The extension-ranking semantics $\sqsupseteq^{\mathsf{cope}(\gamma)}$ can compare all possible sets in $\mathcal{P}(A)$, not just the σ-extensions.

Although the relations $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$ are not *transitive*, $\sqsupseteq^{\mathsf{cope(nonatt)}}$ and $\sqsupseteq^{\mathsf{cope(strdef)}}$ are transitive and reflexive. We can generalise this observation:

Proposition 51. *Let $F = (A, R)$ be an AF. $\sqsupseteq^{\mathsf{cope}(\tau_1,\dots,\tau_n)}_{F}$ is* reflexive *and* transitive *for any sequence of base relations $(\tau_1, \dots, \tau_n)$.* *Proof: P. 190*

While the lexicographic combination lex fully preserves the input relation, when applied to a single base relation, for any abstract argumentation framework $F = (A, R)$ it holds that $\mathsf{lex}(\sqsupseteq^{\tau}_{F}) = \sqsupseteq^{\tau}_{F}$, this behaviour does not always hold for cope.

Example 68. *Consider F_4 from Example 6 and sets $\{b, d\}$ and $\{c, f\}$. Argument b is not attacked by $\{c, f\}$, while d attacks both c and f, meaning:*

$$\{b, d\} \sqsupset^{\mathsf{nonatt}}_{F_4} \{c, f\}$$

However, using cope*, we get:*

$$\{c, f\} \sqsupset^{\mathsf{cope(nonatt)}}_{F_4} \{b, d\}$$

Changing the relationship between $\{b, d\}$ and $\{c, f\}$.

Next, consider the sets $\{b, f\}$ and $\{c, f\}$. Arguments b and f are strongly defended by $\{b, f\}$ against $\{c, f\}$, while c and f are not strongly defended by $\{c, f\}$ against $\{b, f\}$, so:

$$\{b, f\} \sqsupset_{F_4}^{\mathsf{strdef}} \{c, f\}$$

However, using cope*, we get:*

$$\{c, f\} \sqsupset_{F_4}^{\mathsf{cope(strdef)}} \{b, f\}$$

Again altering the relationship between the two sets.

Example 69. *Consider F_5 from Example 22 and the sets $\{h, i\}$ and $\{i, j\}$. These two sets are incomparable with respect to* Conflicts*, i. e.;*

$$\{h, i\} \asymp_{F_5}^{\mathsf{Conflicts}} \{i, j\}$$

However, using cope*, these two sets are comparable and equally plausible:*

$$\{h, i\} \equiv_{F_5}^{\mathsf{cope(Conflicts)}} \{i, j\}$$

Using cope *we can now compare two previously incomparable sets, but the relationship between these two sets has changed.*

The two examples demonstrate that cope is not an identity function for every base relation. We know that $\sqsupseteq^{\mathsf{nonatt}}$, $\sqsupseteq^{\mathsf{strdef}}$, and $\sqsupseteq^{\mathsf{Conflicts}}$ lack important properties, namely *transitivity* ($\sqsupseteq^{\mathsf{nonatt}}$, $\sqsupseteq^{\mathsf{strdef}}$) and *totality* ($\sqsupseteq^{\mathsf{Conflicts}}$). If the underlying base relation τ satisfies these two properties, then cope($\sqsupseteq^{\tau}$) coincides with $\sqsupseteq^{\tau}$.

Proposition 52. *Let $F = (A, R)$ be an AF. If base relation $\sqsupseteq^{\tau}$ is* total *and* transitive*, then* cope($\sqsupseteq^{\tau}_F$) $=\sqsupseteq^{\tau}_F$. *Proof: P. 190*

Next, we explore how the combination method cope complies with our principles when applied to simple sequences of base relations. We begin with cope($\sqsupseteq^{\mathsf{Conflicts}}$, $\sqsupseteq^{\mathsf{Undef}}$). The lexicographic combination of $\sqsupseteq^{\mathsf{Conflicts}}$ and $\sqsupseteq^{\mathsf{Undef}}$ generalises classical admissibility, as shown in Proposition 25. However, lexicographic combination is not the only combination method that generalises admissibility by combining $\sqsupseteq^{\mathsf{Conflicts}}$ and $\sqsupseteq^{\mathsf{Undef}}$. The Copeland-based combination serves the same purpose.

Proposition 53. cope($\sqsupseteq^{\mathsf{Conflicts}}$, $\sqsupseteq^{\mathsf{Undef}}$) *satisfies* ad-generalisation. *Proof: P. 191*

Since ad-*generalisation* is satisfied, cope($\sqsupseteq^{\mathsf{Conflicts}}$, $\sqsupseteq^{\mathsf{Undef}}$) cannot be a generalisation of any other extension semantics.

While cope($\sqsupseteq^{\mathsf{Conflicts}}$, $\sqsupseteq^{\mathsf{Undef}}$) generalises admissibility, *composition* is violated.

Example 70. *Let $F_{19} = (\{a, b, c, d, e, f, g\}, \{(a, b), (b, a), (c, a), (a, c), (b, c), (c, b), (d, d), (d, e), (e, d), (d, f), (f, g)\}$ be an AF, as depicted in Figure 6.17. F_{19} can be split into two*

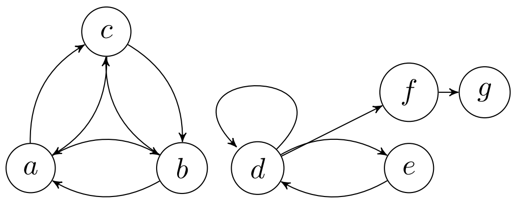

Figure 6.17: AF F_{19} from Example 70.

disjoint AFs $F_{19,1} = (\{a,b,c\}, \{(a,b),(b,a),(a,c),(b,c),(c,b)\})$ *and* $F_{19,2} = (\{d,e,f,g\}, \{(d,d),(d,e),(e,d),(d,f),(f,g)\}$. *Consider the sets* $\{a,d,e\}$ *and* $\{b,e,g\}$*, then we get:*

$$\{a\} \equiv^{\mathsf{cope(Conflicts,Undef)}}_{F_{19,1}} \{b\} \quad \textit{and} \quad \{e,g\} \sqsupset^{\mathsf{cope(Conflicts,Undef)}}_{F_{19,2}} \{d,e\}$$

Thus, $\{b,e,g\}$ *should be at least as plausible to be accepted as* $\{a,d,e\}$ *in* F_{19} *with respect to* $\sqsupseteq^{\mathsf{cope(Conflicts,Undef)}}_{F_{19}}$. *However, we find:*

$$\{a,d,e\} \sqsupset^{\mathsf{cope(Conflicts,Undef)}}_{F_{19}} \{b,e,g\}$$

The set $\{a,d,e\}$ is not conflict-free due to the self-attacking argument d, whereas $\{b,e,g\}$ is conflict-free. Therefore, *respect conflicts* is also violated.

$\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ also violates *decomposition*.

Example 71. *Consider AF* F_{18} *from Example 65, which can be partitioned into* $F_{18,1} = (\{a,b,c\}, \{(a,b),(a,c)\})$ *and* $F_{18,2} = (\{d,e\}, \{(d,e)\})$. *Consider the sets* $\{a,e\}$ *and* $\{c,d\}$. *We have:*

$$\{c,d\} \equiv^{\mathsf{cope(Conflicts,Undef)}}_{F_{18}} \{a,e\} \quad \textit{and} \quad \{d\} \sqsupset^{\mathsf{cope(Conflicts,Undef)}}_{F_{18,2}} \{e\}$$

However, we also have:

$$\{a\} \sqsupset^{\mathsf{cope(Conflicts,Undef)}}_{F_{18,1}} \{c\}$$

Thus, violating decomposition.

Adding a defended argument to a set does not change the number of comparisons the set wins with respect to its conflicts or undefended arguments, thus satisfying *weak reinstatement*.

Proposition 54. $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ *satisfies* weak reinstatement. *Proof: P. 191*

In the proof of Proposition 54, we showed that $E \cup \{a\} \equiv^{\mathsf{cope(Conflicts,Undef)}}_{F} E$ holds, indicating that *strong reinstatement* is violated.

Despite satisfying ad-*generalisation* $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ violates *addition robustness*.

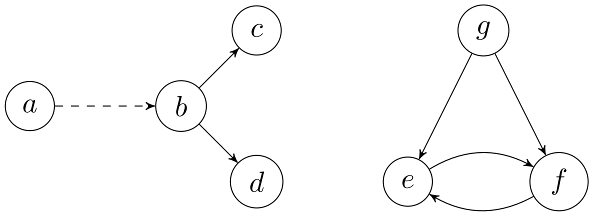

Figure 6.18: AF F_{20} from Example 72, where attack (a, b) is added later to obtain F'_{20}.

Example 72. *Let* $F_{20} = (A_{20}, R_{20}) = (\{a, b, c, d, e, f, g\}, \{(b, c), (b, d), (g, f), (g, e), (f, e), (e, f)\})$ *be an AF, as depicted in Figure 6.18. Consider sets* $\{a, b, c, d\}$ *and* $\{a, e, f\}$. *We have:*

$$\{a, e, f\} \sqsupset_{F_{20}}^{\mathsf{cope(Conflicts,Undef)}} \{a, b, c, d\}$$

Adding an attack between a *and* b *results in AF* $F'_{20} = (A, R \cup \{(a, b)\})$. *If* addition robustness *were satisfied, it would hold that:*

$$\{a, e, f\} \sqsupset_{F'_{20}}^{\mathsf{cope(Conflicts,Undef)}} \{a, b, c, d\}$$

But this is not the case since:

$$\{a, b, c, d\} \sqsupset_{F'_{20}}^{\mathsf{cope(Conflicts,Undef)}} \{a, e, f\}$$

Hence, addition robustness *is violated.*

In Example 68, we saw that $\sqsupseteq^{\mathsf{cope(nonatt)}}$ and $\sqsupseteq^{\mathsf{cope(strdef)}}$ are not identical to $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$, respectively. Therefore, it is interesting to explore the properties of $\mathsf{cope}(\sqsupseteq^{\mathsf{nonatt}})$ and $\mathsf{cope}(\sqsupseteq^{\mathsf{strdef}})$.

Proposition 24 shows that the set containing all arguments is among the most plausible sets for $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$. This suggests that $\mathsf{cope}(\sqsupseteq^{\mathsf{nonatt}})$ and $\mathsf{cope}(\sqsupseteq^{\mathsf{strdef}})$ violate *σ-generalisation* for any conflict-free based extension semantics σ.

Example 73. *Consider* $F_5 = (\{h, i, j\}, \{(h, i), (i, j)\})$ *from Example 22. For both* $\sqsupseteq^{\mathsf{cope(nonatt)}}$ *and* $\sqsupseteq^{\mathsf{cope(strdef)}}$, *the set* $\{h, i, j\}$ *is among the most plausible sets, i. e.,* $\{h, i, j\} \in \mathtt{max}_{\mathsf{cope}(\tau)}(F_5)$ *for* $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$. *Hence,* σ-generalisation *is violated for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

In Example 73, the conflicting set $\{h, i, j\}$ is among the most plausible sets, thus violating *respect conflicts*.

Similar to $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$, $\mathsf{cope}(\sqsupseteq^{\mathsf{nonatt}})$ and $\mathsf{cope}(\sqsupseteq^{\mathsf{strdef}})$ also violate *composition*.

Example 74. *Consider* $F_{21} = (\{a, b, c, d, e\}, \{(a, b), (b, a), (c, d), (c, e)\})$ *as depicted in Figure 6.19.* F_{21} *can be partitioned into two disjoint AFs:* $F_{21,1} = (\{a, b\}, \{(a, b), (b, a)\})$ *and* $F_{21,2} = (\{c, d, e\}, \{(c, d), (c, e)\})$. *Consider the sets* $\{a, c\}$ *and* $\{b, c, d, e\}$. *In* $F_{21,1}$, $\{a\}$ *wins the comparison against* $\emptyset$ *and* $\{b\}$ *with respect to* $\sqsupseteq^{\mathsf{nonatt}}_{F_{21,1}}$ *and* $\sqsupseteq^{\mathsf{strdef}}_{F_{21,1}}$, *whereas*

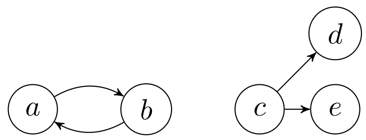

Figure 6.19: AF F_{21} from Example 74.

$\{b\}$ wins the comparison against $\emptyset$ and $\{a\}$, thus $\{a\}$ and $\{b\}$ have the same win/lose record in $F_{21,1}$ with respect to $\sqsupseteq^{\mathsf{cope(nonatt)}}_{F_{21,1}}$ and $\sqsupseteq^{\mathsf{cope(strdef)}}_{F_{21,1}}$. In $F_{21,2}$ similar holds, for $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$, we get:

$$\{a\} \equiv^{\mathsf{cope}(\tau)}_{F_{21,1}} \{b\} \quad \textit{and} \quad \{c\} \equiv^{\mathsf{cope}(\tau)}_{F_{21,2}} \{c, d, e\}$$

In F_{21}, $\{a, c\}$ wins 27 comparisons and loses 15 comparisons with respect to $\sqsupseteq^{\mathsf{nonatt}}_{F_{21}}$ and $\sqsupseteq^{\mathsf{strdef}}_{F_{21}}$, $\{b, c, d, e\}$ wins 27 comparisons and loses 11 comparisons with respect to $\sqsupseteq^{\mathsf{nonatt}}_{F_{21}}$ and $\sqsupseteq^{\mathsf{strdef}}_{F_{21}}$. Thus, $\{b, c, d, e\}$ has a better win/lose ratio than $\{a, c\}$ and therefore for $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$, we get:

$$\{b, c, d, e\} \sqsupset^{\mathsf{cope}(\tau)}_{F_{21}} \{a, c\}$$

This shows that composition *is violated.*

Both $\mathsf{cope}(\sqsupseteq^{\mathsf{nonatt}})$ and $\mathsf{cope}(\sqsupseteq^{\mathsf{strdef}})$ are violating *decomposition*.

Example 75. *Consider $F_{18} = (\{a, b, c, d, e\}, \{(a, b), (a, c), (d, e)\})$ be an AF from Example 18. This AF can be partitioned into $F_{18,1} = (\{a, b, c\}, \{(a, b), (a, c)\})$ and $F_{18,2} = (\{d, e\}, \{(d, e)\})$. Consider the sets $\{a, b, e\}$ and $\{c, d\}$. For $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$, we have:*

$$\{a, b, e\} \sqsupset^{\mathsf{cope}(\sqsupseteq^{\tau})}_{F_{18}} \{c, d\}$$

We also have:

$$\{a, b\} \sqsupset^{\mathsf{cope}(\sqsupseteq^{\tau})}_{F_{18,1}} \{c\} \quad \textit{and} \quad \{d\} \sqsupset^{\mathsf{cope}(\sqsupseteq^{\tau})}_{F_{18,2}} \{e\}$$

Since $\sqsupseteq^{\mathsf{nonatt}}$ and $\sqsupseteq^{\mathsf{strdef}}$ only rely on the attackers, adding a defended argument increases the strength of a set.

Proposition 55. $\mathsf{cope}(\sqsupseteq^{\tau})$ *satisfies* strong reinstatement *for $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$.*

Proof: P. 191

$\mathsf{cope}(\sqsupseteq^{\tau})$ violates *addition robustness* for $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$.

Example 76. *Consider AF $F_{22} = (\{a, b, c, d, e, f\}, \{(c, a), (d, b)\})$ as depicted in Figure 6.20 and the sets $\{a, d, e\}$ and $\{a, b, e, f\}$. These two sets are equally plausible to be accepted with respect to $\mathsf{cope}(\sqsupseteq^{\tau})$ for $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$, i. e.,*

$$\{a, d, e\} \equiv^{\mathsf{cope}(\sqsupseteq^{\tau})}_{F_{22}} \{a, b, e, f\}$$

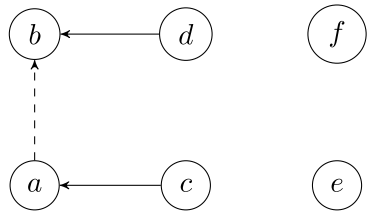

Figure 6.20: AF F_{22} from Example 76, where the attack (a,b) is added later to obtain F'_{22}.

If addition robustness *were satisfied, adding the attack* (a,b) *should not result in:*

$$\{a,b,e,f\} \sqsupset^{\mathsf{cope}(\sqsupseteq^{\tau})}_{F'_{22}} \{a,d,e\}$$

Where $F'_{22} = (\{a,b,c,d,e,f\}, \{(c,a),(d,b)\} \cup \{(a,b)\})$. *However, this is the case for* F'_{22}*, therefore* $\mathsf{cope}(\sqsupseteq^{\tau})$ *violates* addition robustness *for* $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$.

Principles	$\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$	$\mathsf{cope}(\sqsupseteq^{\mathsf{nonatt}})$	$\mathsf{cope}(\sqsupseteq^{\mathsf{strdef}})$
σ-generalisation	✓($\sigma = \mathsf{ad}$)	✗	✗
respect conflicts	✗	✗	✗
composition	✗	✗	✗
decomposition	✗	✗	✗
weak reinstatement	✓	✓	✓
strong reinstatement	✗	✓	✓
addition robustness	✗	✗	✗
syntax independence	✓	✓	✓

Table 6.3: Principles satisfied and violated by $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$, $\mathsf{cope}(\sqsupseteq^{\mathsf{nonatt}})$ and $\mathsf{cope}(\sqsupseteq^{\mathsf{strdef}})$.

It is clear by definition that all extension-ranking semantics discussed in this section satisfy *syntax independence*.

Table 6.3 summarises the results of this section. Throughout this chapter, we have combined previously defined base relations to construct extension-ranking semantics and analysed their behaviour by examining the principles they satisfy. We have demonstrated that when subset-based base relations are combined lexicographically, the resulting extension-ranking semantics satisfies the maximal number of principles. Using the Copeland-based combination allows us to generalise Dung's extension semantics. However, this approach comes at the cost of violating other desired principles.

CONDITIONAL-BASED APPROACHES 7

In abstract argumentation, an attack between two arguments can be interpreted as a conditional statement: "if the attacker is accepted, then the attacked argument cannot be accepted". In logic, such "If A, then B" structures are called *conditionals* and have been extensively studied (Nute, 1980, 1984; Kraus et al., 1990; Lehmann and Magidor, 1992). The relationship between *conditional logic* and argumentation has also been explored in various papers (Heyninck, 2019; Heyninck et al., 2020, 2021b; Kern-Isberner and Simari, 2011; Thimm and Kern-Isberner, 2008; Weydert, 2013, 2014; Skiba and Thimm, 2020; Skiba, 2020b,a). These studies have shown that argumentation models can enhance conditional logics, and integrating conditional logics into argumentation can lead to richer argumentation models.

One approach to reasoning in conditional logic involves selecting the *most plausible worlds*, which represent different truth value assignments to logical atoms. Using *ordinal conditional functions* (OCFs), also known as *ranking functions* (Spohn, 1988), we can rank these assignments and select the one that best fits the scenario. A conditional of the form "If A, then B" can then be interpreted as: if A is assigned `true`, then it is more "believable" to assign `true` to B than `false`.

We can encode attacks of an abstract argumentation framework as conditionals, then a set of arguments represents a plausible assignment. By leveraging ideas from conditional logic and ordinal conditional functions, we can rank sets of arguments. In this chapter, we define an extension-ranking semantics inspired by ordinal conditional functions. For this purpose, we first introduce ordinal conditional functions for abstract argumentation frameworks, and then discuss a particular ordinal conditional function called *z-ranking function* (Goldszmidt and Pearl, 1996), using it to define an extension-ranking semantics.

In the following section, we review the necessary background on conditional logic and ordinal conditional functions, then introduce OCFs for abstract argumentation frameworks. In Section 7.3, we define an extension-ranking semantics and analyse its behaviour.

7.1 BACKGROUND

Before introducing the necessary preliminaries for conditional logic, we need to recall *propositional logic*. For a set of atoms At, we define the *propositional language* $\mathcal{L}(At)$. We use the usual connectives: $\wedge$ (and), $\vee$ (or) and $\neg$ (negation). An *interpretation* (or *possible world*) for $\mathcal{L}(At)$ is defined by a function $\omega : At \rightarrow \{\texttt{true}, \texttt{false}\}$. $\Omega(At)$ denotes the set of all interpretations. An interpretation ω *satisfies* an atom $a \in At$ ($\omega \models a$) if and only if $\omega(a) = \texttt{true}$. We extend $\models$ to formulas as usual.

Conditionals are structures like $(\psi \mid \phi)$, which can be read as "if ϕ then (usually) ψ". An interpretation ω *verifies* a conditional $(\psi \mid \phi)$ if and only if ω satisfies both the antecedent ϕ and the conclusion ψ, i. e., $\omega(\phi) = \texttt{true}$ and $\omega(\psi) = \texttt{true}$. ω *falsifies* the conditional if and only if the antecedent ϕ is satisfied and the conclusion is not, i. e., $\omega(\phi) = \texttt{true}$ and $\omega(\psi) = \texttt{false}$. Otherwise, the conditional is *not applicable*.

In the literature, we can find a number of different semantics for conditionals (Stalnaker, 1968; Lewis, 1973; Nute, 1984; Kraus et al., 1990; Lehmann and Magidor, 1992). In this work, we employ *ordinal conditional functions* (OCFs), also known as *ranking functions* (Spohn, 1988) to reason with conditionals. These OCFs are a *preferential model semantics* (Kraus et al., 1990; Lehmann and Magidor, 1992), which give us a order over the set of interpretations. An ordinal conditional function $\kappa : \Omega(At) \rightarrow \mathbb{N} \cup \{\infty\}$ denotes the degree of plausibility of the interpretation. We define $\kappa(\phi)$ as:

$$\kappa(\phi) := \min\{\kappa(\omega) \mid \omega \models \phi\} \quad \text{with } \min\emptyset = -1$$

An ordinal conditional function *satisfies* a set of conditionals Δ if for each conditional $(\psi \mid \phi) \in \Delta$, $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$. In other words, verifying a conditional is more plausible than falsifying it.

Since there are many ways to define a satisfying ordinal conditional function, this thesis focuses on an approach called *system Z* (Goldszmidt and Pearl, 1996).

Definition 57. *A conditional $\delta \in \Delta$ is* tolerated *by a finite set of conditionals Δ if there exists an interpretation ω, which verifies δ and does not falsify any conditional $\delta' \in \Delta, \delta' \neq \delta$.*

The Z-Partitioning $(\Delta_0, \ldots, \Delta_n)$ *of Δ is defined as:*

$$\Delta_0 = \{\delta \in \Delta \mid \Delta \textit{ tolerates } \delta\}$$
$$\Delta_{1,\ldots,n} \textit{ is the Z-Partitioning of } \Delta \setminus \Delta_0$$

For $\delta \in \Delta$, we define: $Z_\Delta(\delta) = i$ if and only if $\delta \in \Delta_i$ and $(\Delta_0, \ldots, \Delta_n)$ is a Z-partitioning of Δ. We define the ordinal conditional function $\kappa^Z_\Delta(\omega)$ via:

$$\kappa^Z_\Delta(\omega) = \max\{Z_\Delta(\delta) \mid \omega \text{ falsifies } \delta, \delta \in \Delta\} + 1$$

with $\max\emptyset = -1$.

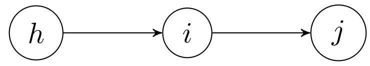

Figure 7.1: Recalled AF F_5 from Example 22.

In other words, the function κ^Z_Δ assigns an interpretation ω a numerical value based on the highest-ranked conditional ω it falsifies. This is well-defined as shown by Goldszmidt and Pearl (1996).

Example 77 (Goldszmidt and Pearl (1996))**.** *Consider the following set of conditionals $\Delta = \{\delta_1, \delta_2, \delta_3\}$ about the flying ability of penguins.*

δ_1*: "birds fly"* $(f \mid b)$ $\quad$ δ_2*: "penguins are birds"* $(b \mid p)$

δ_3*: "penguins do not fly"* $(\neg f \mid p)$

δ_1 *can be tolerated by all conditionals. For* δ_2 *to be tolerated it has to hold that* $\omega(p) = \omega(b) =$ *true, however then* δ_1 *implies* $\omega(f) =$ *true and* δ_3 *implies* $\omega(\neg f) =$ *true. Thus,* δ_2 *is not tolerated by* Δ*. The same holds for* δ_3*. So, the Z-Partitioning of* Δ *is* $\Delta_0 = \{\delta_1\}$ *and* $\Delta_1 = \{\delta_2, \delta_3\}$*. We calculate the plausibility degree of an interpretation* ω*. For example, a flying penguin (*$\omega(p) = \omega(b) = \omega(f) =$ `true`*) receives a value of* $\kappa^Z_\Delta(\omega) = 2$*, since* δ_3 *is falsified.*

7.2 ORDINAL CONDITIONAL FUNCTIONS FOR ABSTRACT ARGUMENTATION

In propositional logic, interpretations evaluate an atom as either `true` or `false`, similar to how extension semantics in abstract argumentation classify sets of arguments as accepted or not. In conditional logic, interpretations are ranked based on their plausibility. Thus, ordinal conditional functions relate to propositional logic in the same way that extension-ranking semantics relate to extension semantics. We define ordinal conditional functions for abstract argumentation frameworks. An ordinal conditional function $\kappa(I)$ for an abstract argumentation framework $F = (A, R)$ is used to compute a numerical plausibility value for a set of arguments $I \subseteq A$ to be considered acceptable.

Definition 58. *Let* $F = (A, R)$ *be an AF, an* ordinal conditional function *is a function* $\kappa : \mathcal{P}(A) \to \mathbb{N} \cup \{\infty\}$ *with* $\kappa^{-1}(0) \neq \emptyset$*.*

Example 78. *Consider AF* F_5 *from Example 22 (recalled in Figure 7.1) and an example ordinal conditional function* κ*.* $\{h, j\}$ *is the preferred extension, so it receives the best possible value of* 0*, since there is no set more plausible to be accepted, i. e.* $\kappa(\{h, j\}) = 0$*. The two admissible sets* $\{h\}$ *and* $\emptyset$ *receive a value of* 1 *(*$\kappa(\{h\}) = 1$ *and* $\kappa(\emptyset) = 1$*). The conflict-free sets* $\{i\}$ *and* $\{j\}$ *receive a value of* 2*,* $\kappa(\{i\}) = 2$ *and* $\kappa(\{j\}) = 2$*. Since* $\{h, i\}$ *and* $\{i, j\}$ *are involved in only one conflict, they receive a better value than* $\{h, i, j\}$*, so* $\kappa(\{h, i\}) = \kappa(\{j, i\}) = 3$ *and* $\kappa(\{h, i, j\}) = 4$*.*

Conditional logic adheres to the principle that "a conditional is satisfied if its verification is more plausible than its violation". For abstract argumentation frameworks the two guiding principles are *Conflict-freeness* and *Reinstatement*.

Conflict-freeness: An argument should not be part of an acceptable set if one of its attackers is part of the acceptable set.

Reinstatement: An argument should be part of an acceptable set if all its attackers are not part of an acceptable set.

Conflict-freeness ensures that an acceptable set of arguments does not contain conflicting arguments, making conflicting sets less plausible to be accepted than conflict-free sets. *Reinstatement* suggests that if there is no reason to reject an argument, it should not be rejected. A set that defends itself against all attackers is at least as plausible as a set that does not. We implement these two guiding principles in the ordinal conditional function κ.

Definition 59. *Let $F = (A, R)$ be an AF, $a, b \in A$, and κ a ordinal conditional function:*

- κ accepts *an attack (a, b) with $a \neq b$ if $\kappa(\{a\}) < \kappa(\{a, b\})$.*
- κ possibly reinstates *an argument a if $\kappa(S \cup \{a\}) \leq \kappa(S)$ for all $S \subseteq A$ with $S \cap (a_F^- \cup a_F^+) = \emptyset$.*

In other words, for an attack (a, b) to be accepted by an ordinal conditional function, it is more plausible that a is part of an acceptable set than that both a and b are part of an acceptable set. An argument a is possibly reinstated by an ordinal conditional function if, for all sets S that do not conflict with a, it is at least as plausible that S and a together are part of an acceptable set as S alone.

If an ordinal conditional function adheres to the principles of conflict-freeness and reinstatement for all arguments and attacks within an abstract argumentation framework, then it is said to satisfy that AF.

Definition 60. *An ordinal conditional function κ* satisfies *an AF $F = (A, R)$ if it accepts all attacks in R and possibly reinstates all arguments in A.*

Example 79. *Consider F_5 from Example 22 and an ordinal conditional function κ. For κ to satisfy F_5, it must accept the two attacks, meaning:*

$$\kappa(\{h\}) < \kappa(\{h, i\}) \quad \textit{and} \quad \kappa(\{i\}) < \kappa(\{i, j\})$$

Additionally, κ must possibly reinstate all arguments. For h, we have:

$$\kappa(\{h\}) \leq \kappa(\emptyset) \quad \textit{and} \quad \kappa(\{h, j\}) \leq \kappa(\{j\})$$

For i and j, we get:

$$\begin{aligned} &\kappa(\{i\}) \leq \kappa(\emptyset) \\ &\kappa(\{j\},) \leq \kappa(\emptyset) \quad \textit{and} \quad \kappa(\{h, j\}) \leq \kappa(\{h\}) \end{aligned}$$

Table 7.1 depicts an ordinal conditional function κ that satisfies F_5. The complete set $\{h, j\}$ receives a value of 0, indicating it is the most plausible set. This function allows the comparison of non-admissible sets, such as $\{i\}$ and $\{h, i\}$, where $\{i\}$ receives a value of 3 and $\{h, i\}$ a value if 4, making $\{i\}$ more plausible than $\{h, i\}$.

i	$\kappa^{-1}(i)$
6	$\{h,i,j\}$, $\{i,j\}$
5	$\{h,i\}$
4	$\emptyset$
3	$\{i\}$
2	$\{j\}$
1	$\{h\}$
0	$\{h,j\}$

Table 7.1: Example ordinal conditional function κ for Example 79.

Note that if an abstract argumentation framework contains a self-attacking argument a, no ordinal conditional function can satisfy that AF. Since, accepting the attack (a,a) would require $\kappa(\{a\}) < \kappa(\{a\})$, which is impossible for any function κ.

With the foundational concepts for ordinal conditional functions in abstract argumentation, we can now explore a function inspired by System Z. Recall that in System Z, a conditional $(\phi|\psi)$ is *tolerated* by a set of conditionals if it is confirmed by an interpretation ω without refuting any other conditional. For an attack (a,b), we say "if a is part of an acceptable set then b is not part of the set". Thus, we interpret the set of attacks as a set of conditionals and seek a set of arguments (interpretation) that *tolerates* an attack without violating any other.

Definition 61. *Let $F = (A,R)$ be an AF and $a,b \in A$ such that $(a,b) \in R$:*

- *A set $E \subseteq A$* verifies *an attack (a,b) if and only if $a \in E$ and $b \notin E$.*
- *A set $E \subseteq A$* violates *an attack (a,b) if and only if $a \in E$ and $b \in E$.*
- *A set $E \subseteq A$* satisfies *an attack (a,b) if and only if (a,b) is not violated.*

In other words, a set of arguments satisfies an attack if it does not contain both the attacker and the target of the attack.

Example 80. *Consider F_5 from Example 22. The set $\{h\}$ verifies the attack (h,i) and does not violate the attack (i,j) since $i \notin \{h\}$. The set $\{h,i\}$ verifies attack (i,j), however, $\{h,i\}$ violates the attack (h,i).*

Verifying an attack in abstract argumentation captures only the aspect of *conflict-freeness*. To fully capture reasoning in abstract argumentation, we need to model *reinstatement* as well. For this, we introduce an additional condition known as *attack admissibility*. This condition ensures that if all attackers of an argument are excluded from a set, then the argument itself should be included in the set.

Definition 62. *Let $F = (A,R)$ be an AF and an argument $a \in A$.*

- *A set $E \subseteq A$* verifies attack admissibility *of a if and only if $a \in E$ and $b \notin E$ for all $b \in a^-_F$.*

- *A set $E \subseteq A$* violates attack admissibility *of a if and only if $a \notin E$ and $b \notin E$ for all $b \in a_F^-$.*
- *A set $E \subseteq A$* satisfies attack admissibility *of a if and only if E does not violate attack admissibility of a.*

Example 81. *Consider F_5 from Example 22. The set $\{h, j\}$ verifies attack admissibility of argument j because the only attacker of j, which is i, is not in $\{h, j\}$ and one of the attackers of i is containing in $\{h, j\}$. For $\{i\}$, we know that h is not part of the set, but $\{i\}$ does not attack h, so attack admissibility of h is violated.*

By combining the notions of verifying an attack and attack admissibility, we can define when an attack can be *tolerated*:

Definition 63. *Let $F = (A, R)$ be an AF and arguments $a, b \in A$. A set $R' \subseteq R$* tolerates *an attack (a, b) if and only if there is a set $E \subseteq A$ that:*

1. *verifies (a, b),*
2. *satisfies each attack in R', and*
3. *satisfies attack admissibility of each $c \in A$.*

To tolerate an attack, we must find a set of arguments E that contains the attacker, is conflict-free, and every argument not in E must be attacked.

Example 82. *Consider F_5 from Example 22. The attack (i, j) is not tolerated by $\{(h, i), (i, j)\}$. For (i, j) to be verified by any set $E \subseteq \{h, i, j\}$, it must hold that $i \in E$. Then, to not violate (h, i), we must have $h \notin E$. However, E would then not contain any attacker of h, violating attack admissibility of h.*

With the notion of toleration, we can define an ordinal condition function inspired by System Z for abstract argumentation frameworks.

Definition 64. *Let $F = (A, R)$ be an AF and the Z-attack-Partitioning $(R_0, \ldots, R_n)$ with $R_0 \cup \ldots \cup R_n \subseteq R$ is defined as:*

- $R_0 = \{r \in R \mid R \text{ tolerates } r\}$
- *$(R_1, \ldots, R_n)$ is the Z-attack-Partitioning of $R \setminus R_0$*

For $r \in R$, define $Z_R(r) = i$ if $r \in R_i$. The ordinal conditional function $\kappa_F^Z(E)$ for $E \subseteq A$ is defined as:

$$\kappa_F^Z(E) = \max\{Z_R(r) \mid E \text{ violates } r\} + 1$$

Attacks in R_0 are tolerated with respect to the entire set of attacks R of an abstract argumentation framework $F = (A, R)$, while attacks in R_1 are tolerated after removing the attacks in R_0. Using this partitioning of attacks, we can rank sets of arguments based on their plausibility of acceptance with respect to the attacks. If a set

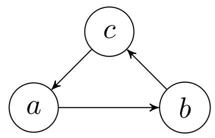

Figure 7.2: Recalled AF F_7 from Example 27.

violates an attack at level 0, while another set violates an attack at level 1, then the first set is more plausible to accept than the second. The higher an attack is ranked, the worse its violation. Thus, the partitioning of attacks can be interpreted as a ranking based on the impact of each attack in the abstract argumentation framework, with lower-ranked attacks being more significant. It is more important to satisfy a single high-ranked attack than to satisfy several low-ranked attacks.

Example 83. *Consider F_5 from Example 22. In Example 82, we have discussed that attack (i, j) is not tolerated by $\{(h, i), (i, j)\}$, so (i, j) is not part of R_0 for the Z-attack-Partitioning of the attacks of F_5. The attack (h, i) is verified by the set $\{h, j\}$. This set also satisfies (i, j) and satisfies attack admissibility for h, i, and j. Hence, (h, i) is tolerated. We can define the Z-attack-Partitioning with:*

$$R_0 = \{(h, i)\}$$
$$R_1 = \{(i, j)\}$$

Next, we compute $\kappa^Z_{F_5}$. We get $\kappa^Z_{F_5}(\{h, i\}) = 2$, since $\{h, i\}$ violates the attack (h, i) in R_0. For $\{h, i, j\}$ we get $\kappa^Z_{F_5}(\{h, i, j\}) = 3$.

We will now discuss some properties of κ^Z_F. We start by showing that κ^Z_F actually satisfies the abstract argumentation framework F.

Proposition 56. *If κ^Z_F is defined, then κ^Z_F satisfies AF F.* *Proof: P. 192*

Besides being undefined for abstract argumentation frameworks with self-attacks, κ^Z is also undefined for AFs without a stable extension.

Example 84. *Consider F_7 from Example 27 recalled in Figure 7.2. If we want to tolerate (a, b) by $\{(a, b), (b, c), (c, a)\}$, we need to find a set $E \subseteq \{a, b, c\}$ that verifies (a, b) and satisfies every other attack and satisfies attack admissibility for every argument. Since we need to verify (a, b), we have $a \in E$. But this means that $b, c \notin E$, otherwise E is not conflict-free. This violates attack admissibility of c. Similarly, we can show that (b, c) and (c, a) cannot be tolerated either. We cannot define a Z-attack-Partitioning for F_7.*

Next, we show that if an abstract argumentation framework F has no stable extension, then κ^Z_F is undefined.

Proposition 57. *κ^Z_F is undefined for AF F with $\mathsf{st}(F) = \emptyset$.* *Proof: P. 192*

Examining the levels of a Z-attack-Partitioning in detail, we see that, if an attack (a, b) is in R_0, then a is credulously accepted with respect to admissibility in F.

Proposition 58. *For AF $F = (A, R)$ and Z-attack-Partitioning $(R_0, \ldots, R_n)$ of R, if $(a, b) \in R_0$ then $a \in \mathsf{cred}_{\mathsf{ad}}(F)$.* *Proof: P. 193*

$$
\begin{tikzcd}
 & \{i,j\} \arrow[rr, dash] \arrow[dr] & & \{h,i,j\} \arrow[dl] & \\
 & & \{h,i\} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\emptyset \arrow[r, dash] & \{h\} \arrow[r, dash] & \{i\} \arrow[r, dash] & \{j\} \arrow[r, dash] & \{h,j\}
\end{tikzcd}
$$

Figure 7.3: Lattice depicting $\sqsupseteq^{\kappa^Z}_{F_5}$ from Example 85.

7.3 EXTENSION-RANKING SEMANTICS BASED ON SYSTEM Z

Next, we define an extension-ranking semantics based on ordinal conditional functions for abstract argumentation frameworks, as discussed in the previous section. A set of arguments E is at least as plausible to be accepted as another set E' if an ordinal conditional function returns a lower value for E than for E'.

Definition 65. *Let $F = (A, R)$ be an AF, sets $E, E' \subseteq A$, and κ an ordinal conditional function satisfying F, the* ordinal conditional-based extension-ranking semantics $\sqsupseteq^{\kappa}_{F}$ *is defined as:*

$$E \sqsupseteq^{\kappa}_{F} E' \textit{ if and only if } \kappa(E) \leq \kappa(E')$$

In other words, E is at least as plausible to be accepted as E' if it is more plausible for E to be in accepted than for is E' to be accepted with respect to an ordinal conditional function κ.

In the following discussion, we focus on the ordinal conditional function κ^Z, but other OCFs, such as *C-representation* (Kern-Isberner, 2001), can be found in the literature.

Example 85. *Consider AF F_5 from Example 22. Recall the corresponding Z-attack-Partitioning from Example 83:*

$$R_0 = \{(h, i)\}$$
$$R_1 = \{(i, j)\}$$

Conflict-free sets are the most plausible sets with respect to $\sqsupseteq^{\kappa^Z}_{F_5}$, while the sets with conflicts are ranked lower. Although the set $\{h, i, j\}$ has more conflicts than $\{i, j\}$, these two sets are equally as plausible to be accepted with respect to $\sqsupseteq^{\kappa^Z}_{F_5}$, since both these sets are violating the attack (i, j). The full extension ranking $\sqsupseteq^{\kappa^Z}_{F_5}$ is depicted in Figure 7.3.

Next, we examine the principles that $\sqsupseteq^{\kappa^Z}$ satisfies. In Example 85, all conflict-free sets are among the most plausible sets, and in particular set $\{i\} \in \mathtt{max}_{\sqsupseteq^{\kappa^Z}}(F_5)$. Thus, *σ-generalisation* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ is violated by $\sqsupseteq^{\kappa^Z}$. However, because all conflict-free sets are among the most plausible sets $\sqsupseteq^{\kappa^Z}$ satisfies *respect conflicts* and *σ-completeness* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

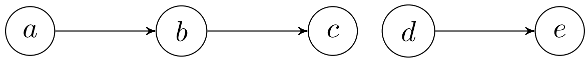

Figure 7.4: AF F_{23} from Example 86.

Principles	$\sqsupseteq^{\kappa^Z}$
σ-generalisation	✗
respect conflicts	✓
composition	✓
decomposition	✗
weak reinstatement	✓
strong reinstatement	✗
addition robustness	✓
syntax independence	✓

Table 7.2: Principles satisfied and violated by $\sqsupseteq^{\kappa^Z}$.

Proposition 59. $\sqsupseteq^{\kappa^Z}$ *satisfies* respect conflicts *and* σ-completeness *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$. *Proof: P. 193*

Next, we look at *composition* and *decomposition*.

Proposition 60. $\sqsupseteq^{\kappa^Z}$ *satisfies* composition. *Proof: P. 193*

While *composition* is satisfied, *decomposition* is violated.

Example 86. *Let* $F_{23} = (\{a, b, c, d, e\}, \{(a, b), (b, c), (d, e)\})$ *This AF can be split into two disjoint AFs:* $F_{23.1} = (\{a, b, c\}, \{(a, b), (b, c)\})$ *and* $F_{23.2} = (\{d, e\}, \{(d, e)\})$. *The Z-attack-Partitioning of* R_{23} *is* $R_{23\ 0} = \{(a, b), (d, e)\}$ *and* $R_{23\ 1} = \{(b, c)\}$. *Let* $\{a, b, d, e\}$ *and* $\{b, c, d\}$*, then* $\kappa^Z_{F_{23}}(\{a, b, d, e\}) = 1$ *and* $\kappa^Z_{F_{23}}(\{b, c, d\}) = 2$. *But,* $\kappa^Z_{F_{23}}(\{a, b, d, e\} \cap A_{23.2} = 1$ *and* $\kappa^Z_{F_{23}}(\{b, c, d\} \cap A_{23.2}) = 0$*, meaning that* decomposition *is violated.*

For *weak reinstatement*, we show that κ^Z satisfies this principle.

Proposition 61. $\sqsupseteq^{\kappa^Z}$ *satisfies* weak reinstatement. *Proof: P. 194*

In Example 85, we have: $\{h\} \equiv^{\kappa^Z}_{F_5} \{h, j\}$, however argument j is defended by $\{h\}$, showing that *strong reinstatement* is violated.

Since only the worst ranked attack violated by a set E is relevant for computing $\kappa^Z(E)$, we can show that *addition robustness* is satisfied by $\sqsupseteq^{\kappa^Z}$.

Proposition 62. $\sqsupseteq^{\kappa^Z}$ *satisfies* addition robustness. *Proof: P. 194*

It is clear by definition that $\sqsupseteq^{\kappa^Z}$ satisfies *syntax independence*.

In this chapter, we introduced the extension-ranking semantics κ^Z, which is based on conditional logics. Table 7.2 shows the satisfied and violated principles of κ^Z. Although κ^Z does not generalising Dung's extension semantics, it enables the comparison of two non-acceptable sets of arguments and satisfies a good number of principles of well-behaved extension-ranking semantics.

This chapter also incorporates recent work on the relationship between conditional logic and argumentation. It demonstrates that both fields can benefit from each other. Conditional logic provides effective methods for ranking sets of arguments. While we focused on how argumentation can benefit from conditional logic, the reverse, how conditional logic can benefit from argumentation, has also been explored in the literature (Heyninck, 2019; Heyninck et al., 2020, 2021b; Kern-Isberner and Simari, 2011; Thimm and Kern-Isberner, 2008; Weydert, 2013, 2014).

LIFTING OPERATORS IN ABSTRACT ARGUMENTATION 8

In the previous chapters, we discussed sets of arguments as a single unit, every argument of a set was considered equally important for the acceptance of the set. However, such an assumption is not always recommended. Unattacked arguments are considered as the strongest arguments, they are part of every complete extension, while arguments with incoming attacks are depended on their defenders to be accepted. Hence, the strength of the individual arguments should not be neglected. Argument-ranking semantics focus on individual arguments, determining their strength to establish a ranking. In this chapter, we utilise argument-ranking semantics to propose a family of extension-ranking semantics.

Both argument-ranking and extension-ranking semantics share the goal of ranking elements, while argument-ranking semantics rank individual objects, extension-ranking semantics rank sets of objects. This chapter explores the relationship between these two types of rankings, which has been a significant topic in *computational social choice* (Brandt et al., 2016b; Rothe et al., 2015; Barberà et al., 2004; Maly, 2020; Moretti and Öztürk, 2017; Bernardi et al., 2019; Khani et al., 2019; Haret et al., 2018a; Suzuki and Horita, 2024). We will explore the connections between extension-rankings and argument-ranking semantics by translating one into the other. This chapter begins by translating argument rankings into extension rankings using *lifting operators* (for an overview see Barberà et al. (2004) and Maly (2020)). In Chapter 9, we examine the reverse process, known as *social rankings*, which translates a ranking over sets of objects into a ranking over objects.

In computational social choice, lifting operators are used to find the "best" committee, such as a group of politicians or a sports team, based on individual performance. For example, a football team manager selecting the starting line-up considers each player's recent performance and chooses those who best fit the game plan. This selection can be done in various ways, such as picking the top performers or creating a balanced team.

We will use lifting operators to define a new extension-ranking semantics. We start with a general definition of a lifting operator in the context of abstract argumentation frameworks.

Definition 66. *Let $F = (A, R)$ be an AF, a* lifting operator ι *takes a preorder $\succeq^{\rho}$ over $E \subseteq A$ as input and outputs an extension-ranking $\sqsupseteq_F^{\iota(\rho)}$.*

Given a preorder over the set of arguments (typically an argument ranking), we construct an extension ranking.

Next, we will discuss various lifting operators. We will start with those previously defined for abstract argumentation frameworks but not yet explored as lifting operators (Sections 8.1, 8.2, 8.3). Then, we will examine lifting operators introduced in computational social choice and discuss their application in formal argumentation (Section 8.4).

8.1 ARGUMENT QUALITY COUNT

Argument-ranking semantics assess the strength of each argument within an abstract argumentation framework. A set containing more highly ranked arguments should be considered more plausible to be accepted than a set with lower-ranked arguments. Bonzon et al. (2018) suggested using argument-ranking semantics to compare σ-extensions, they define what constitutes as a "best" σ-extension. We generalise their approach to compare arbitrary sets of arguments. We begin by defining a base relation where a set E is more plausible to be accepted than E' if E contains more higher-ranked arguments than E'.

Definition 67. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics. For sets $E, E' \subseteq A$, we define the* argument quality count $\mathsf{Aqc}_{\rho,F}(E, E')$ *between E and E' with respect to ρ as:*

$$\mathsf{Aqc}_{\rho,F}(E, E') = |\{(a, b) \text{ such that } a \succ_F^{\rho} b \text{ with } a \in E \text{ and } b \in E'\}|$$

The corresponding base relation $\sqsupseteq_F^{\mathsf{Aqc}_\rho}$ via:

$$E \sqsupseteq_F^{\mathsf{Aqc}_\rho} E' \text{ if and only if } \mathsf{Aqc}_{\rho,F}(E, E') \geq \mathsf{Aqc}_{\rho,F}(E', E)$$

In other words, the argument quality count is the number of arguments of E that are ranked better than arguments in E' with respect to an argument-ranking semantics ρ.

Example 87. *Consider again F_4 from Example 6, depicted in Figure 8.1. The corresponding argument ranking according to the h-categoriser argument-ranking semantics as shown in Example 10 is:*

$$a \succ_{F_4}^{Cat} g \succ_{F_4}^{Cat} d \succ_{F_4}^{Cat} e \succ_{F_4}^{Cat} b \succ_{F_4}^{Cat} c \succ_{F_4}^{Cat} f$$

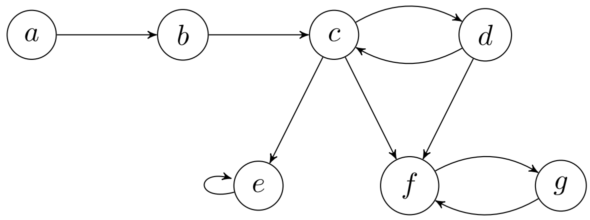

Figure 8.1: Recalled AF F_4 from Example 6.

Consider the sets $\{a, c, g\}$ and $\{a, b, g\}$. The first set is a stable extension, while the second is not conflict-free. The highest-ranked argument is a, so $a \succ_{F_4}^{Cat} b$, $a \succ_{F_4}^{Cat} g$, and $a \succ_{F_4}^{Cat} c$. Also, g is ranked better than b and c, hence $\mathsf{Aqc}_{Cat,F_4}(\{a, c, g\}, \{a, b, g\}) = 3$. $\mathsf{Aqc}_{Cat,F_4}(\{a, b, g\}, \{a, c, g\}) = 4$ because $b \succ_{F_4}^{Cat} c$, and therefore:

$$\{a, b, g\} \sqsupset_{F_4}^{\mathsf{Aqc}_{Cat}} \{a, c, g\}$$

Even though $\{a, c, g\}$ is a stable extension, it is ranked lower than the conflicting set $\{a, b, g\}$.

Example 87 shows that $\sqsupseteq^{\mathsf{Aqc}_\rho}$ ignores internal conflicts within sets. Fortunately, the set containing all arguments A is not always among the most plausible sets.

Example 88. *Continuing from Example 87, consider sets $\{a\}$ and $\{a, b, c, d, e, f, g\}$. Argument a is ranked better than every other argument according to the h-categoriser argument-ranking semantics, thus $\mathsf{Aqc}_{Cat,F_4}(\{a\}, \{a, b, c, d, e, f, g\}) = 6$ and $\mathsf{Aqc}_{Cat,F_4}(\{a, b, c, d, e, f, g\}, \{a\}) = 0$. Therefore we get:*

$$\{a\} \sqsupset_{F_4}^{\mathsf{Aqc}_{Cat}} \{a, b, c, d, e, f, g\}$$

In addition to ignoring internal conflicts, $\sqsupseteq^{\mathsf{Aqc}_\rho}$ is not transitive.

Example 89. *Again, continuing from Example 87, consider the sets $\{a, f\}$, $\{d, g\}$, and $\{a, c\}$. Then we get:*

$$\{a, f\} \equiv_{F_4}^{\mathsf{Aqc}_{Cat}} \{d, g\} \quad \textit{and} \quad \{a, c\} \equiv_{F_4}^{\mathsf{Aqc}_{Cat}} \{d, g\}$$

But it holds that:

$$\{a, c\} \sqsupset_{F_4}^{\mathsf{Aqc}_{Cat}} \{a, f\}$$

This means $\sqsupseteq_{F_4}^{\mathsf{Aqc}_{Cat}}$ is not transitive.

Since $\sqsupseteq^{\mathsf{Aqc}_\rho}$ is not transitive, it is not a lifting operator, as lifting operators induce preorders. However, as shown in Proposition 51, we can use the Copeland-based combination to induce a transitive relation based on $\sqsupseteq_F^{\mathsf{Aqc}_\rho}$, thus creating an extension-ranking semantics.

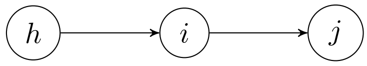

Figure 8.2: Recalled AF F_5 from Example 22.

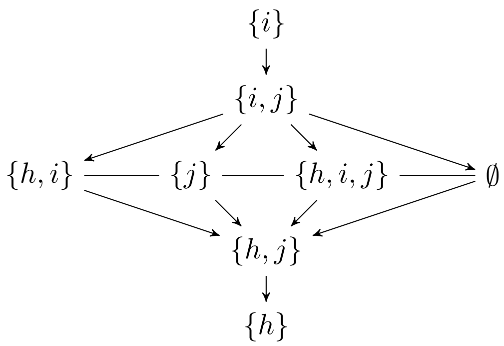

Figure 8.3: Lattice depicting $\sqsupseteq^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})}_{F_5}$ from Example 90.

Example 90. *Consider F_5 from Example 22 recalled in Figure 8.2. The corresponding argument ranking according to the h-categoriser argument-ranking semantics is:*

$$h \succ^{Cat}_{F_5} j \succ^{Cat}_{F_5} i$$

The set $\{h\}$ is the most plausible set for $\sqsupseteq^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})}_{F_5}$. This extension-ranking semantics behaves differently to the previously discussed semantics, since the set $\{i\}$ is the worst ranked set, even though that set is conflict-free. The set containing every argument $\{h, i, j\}$ is equally plausible to be accepted as the empty set. Figure 8.3 shows the induced extension ranking.

Next, we examine the behaviour of $\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})$ according to the principles we have introduced. In Example 90, we see that the stable set $\{h, j\}$ is not one of the most plausible sets, this shows that $\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})$ violates *σ-generalisation* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$. Additionally, since $\{h, i, j\} \sqsupset^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})}_{F_5} \{i\}$, the principle *respect conflicts* is also violated. These observations hold for any argument-ranking semantics that induce the same argument ranking for F_5, such as the *burden-based argument-ranking semantics* and the *serialisability argument-ranking semantics*. To show the violation of *composition* and *decomposition*, we revisit Examples 74 and 65.

Example 91. *Consider F_{21} from Example 74, recalled in Figure 8.4, and the sets $\{a, c\}$ and $\{a, b, c\}$. We have:*

$$\{a\} \equiv^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})}_{F_{21,1}} \{a, b\} \quad \textit{and} \quad \{c\} \equiv^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})}_{F_{21,2}} \{c\}$$

However, we have:

$$\{a, c\} \sqsupset^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})}_{F_{21}} \{a, b, c\}$$

Thus, composition *is violated.*

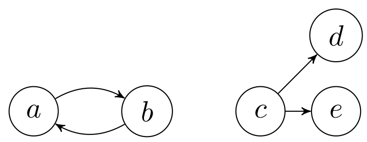

Figure 8.4: Recalled AF F_{21} from Example 74.

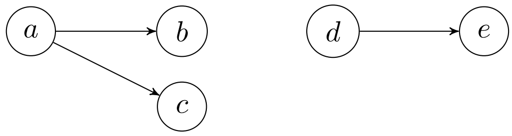

Figure 8.5: Recalled AF F_{18} from Example 65.

Example 92. *Consider F_{18} from Example 65, depicted in Figure 8.5, and the sets $\{a, b, e\}$ and $\{c, d\}$. We have:*

$$\{c, d\} \sqsupset_{F_{18}}^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})} \{a, b, e\} \quad \textit{and} \quad \{d\} \sqsupset_{F_{18,2}}^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})} \{e\}$$

However:

$$\{a, b\} \sqsupset_{F_{18,1}}^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})} \{c\}$$

Thus, decomposition *is violated.*

Unlike other extension-ranking semantics based on cope, $\mathsf{cope}(\sqsupseteq_F^{\mathsf{Aqc}_{Cat}})$ violates *weak reinstatement* and, consequently, *strong reinstatement*.

Example 93. *Continuing from Example 90, consider the set $\{h\}$. The argument j is defended by h and can be reinstated. It should hold that:*

$$\{h, j\} \sqsupseteq_{F_5}^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})} \{h\}$$

But this is not the case, so weak reinstatement *is violated.*

Addition robustness is also violated by $\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})$.

Example 94. *Let $F_{24} = (\{a, b, c, d, e, f\}, \{(b, c), (b, e), (c, e), (d, e)\})$ be an AF as depicted in Figure 8.6. Consider the two sets $\{a, e\}$ and $\{b, c, d, f\}$. For these two sets:*

$$\{a, e\} \equiv_{F_{24}}^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})} \{b, c, d, f\}$$

We can add the attack (a, b) and create $F'_{24} = (\{a, b, c, d, e, f\}, \{(b, c), (b, e), (c, e), (d, e)\} \cup \{(a, b)\})$. However, in F'_{24}, we have:

$$\{b, c, d, f\} \sqsupset_{F'_{24}}^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})} \{a, e\}$$

Thus, addition robustness *is violated.*

Since the names of the arguments do not affect the *h-categoriser argument-ranking semantics*, *syntax independence* is satisfied by $\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})$. Table 8.1 summarises the results of this section, showing that $\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})$ only satisfies *syntax independence*.

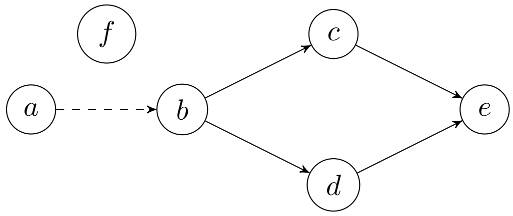

Figure 8.6: AF F_{24} from Example 94, where attack (a, b) is added later to obtain F'_{24}.

Principles	$\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})$
σ-generalisation	✗
respect conflicts	✗
composition	✗
decomposition	✗
weak reinstatement	✗
strong reinstatement	✗
addition robustness	✗
syntax independence	✓

Table 8.1: Principles satisfied and violated by $\mathsf{cope}(\mathsf{Aqc}_{Cat})$.

8.2 HOARE'S LIFTING OPERATOR

Amgoud and Ben-Naim (2013) already discussed how to compare two sets of attackers to define the *Counter-Transitivity* principles for argument-ranking semantics. To recap, an argument a is at least as strong as argument b if and only if they have the same number of attackers, and for every attacker of a, we can find an attacker of b that is better ranked. Essentially, this involves comparing the sets of attackers of a and b, a concept referred as *group comparison*. In the literature on lifting operators (Barberà et al., 2004; Maly, 2020), we find *Hoare's operator* (Brewka et al., 2010), where a set of objects A is at least as preferred as a set of objects B if, for every object in B, there is an object in A that is at least as preferred. The similarities between Hoare's lifting operator and group comparison are evident. In fact, group comparison is essentially Hoare's lifting operator applied to argument-ranking semantics. While lifting operators and argument-ranking semantics have always been related, they have never been discussed together.

Definition 68. *Let $F = (A, R)$ be an AF, sets $E, E' \subseteq A$, and ρ an argument-ranking semantics. We define* Hoare's extension-ranking semantics (Hoare_ρ) *as follows:*

$$E \sqsupseteq_F^{\mathsf{Hoare}_\rho} E' \text{ if and only if } \forall b \in E' \; \exists a \in E \text{ such that } a \succeq_F^\rho b$$

Informally, a set E is at least as plausible to be accepted as E' if, for every argument of E', there is an argument in E that is ranked at least as good according to a preorder ρ.

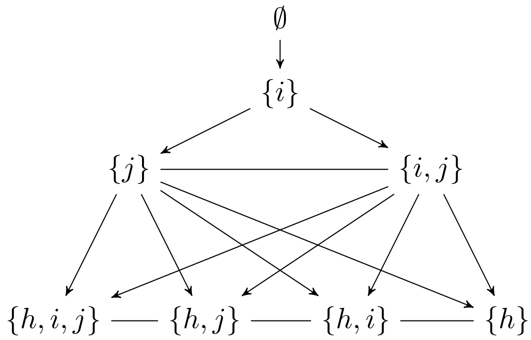

Figure 8.7: Lattice depicting $\sqsupseteq_{F_5}^{\mathsf{Hoare}_{Cat}}$ from Example 96.

Example 95. *Consider AF F_4 from Example 6 and the sets $\{a, b, c\}$ and $\{d, e, g\}$. Using* h-categoriser argument-ranking semantics *as the underlying preorder, we have: $a \succ_{F_4}^{Cat} d$, $a \succ_{F_4}^{Cat} e$, and $a \succ_{F_4}^{Cat} g$. Thus:*

$$\{a, b, c\} \sqsupset_{F_4}^{\mathsf{Hoare}_{Cat}} \{d, e, g\}$$

Comparing this result with $\mathsf{cope}(\sqsupseteq_{F_4}^{\mathsf{Aqc}_{Cat}})$*, we see that* $\mathsf{Aqc}_{Cat,F_4}(\{a, b, c\}, \{d, e, g\}) = 3$ *and* $\mathsf{Aqc}_{Cat,F_4}(\{d, e, g\}, \{a, b, c\}) = 6$*, so:*

$$\{d, e, g\} \sqsupset_{F_4}^{\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})} \{a, b, c\}$$

Thus, these two extension-ranking semantics behave differently.

Example 96. *Consider AF F_5 and the* h-categoriser argument-ranking semantics. *The set $\{h, i, j\}$ is among the most plausible sets with respect to $\sqsupseteq_{F_5}^{\mathsf{Hoare}_{Cat}}$. The two admissible sets $\{h\}$ and $\{h, j\}$ are also among the most plausible sets. The extension ranking is depicted in Figure 8.7.*

Note that $\sqsupseteq^{\mathsf{Hoare}_\rho}$ is reflexive and transitive.

Proposition 63. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics. $\sqsupseteq_F^{\mathsf{Hoare}_\rho}$ is reflexive and transitive.* *Proof: P. 194*

We begin the investigation of the satisfied principles of Hoare_ρ by demonstrating that the set containing all arguments A is among the most plausible sets for any argument-ranking semantics.

Proposition 64. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics, then $A \in \mathtt{max}_{\mathsf{Hoare}_\rho}(F)$.* *Proof: P. 194*

Proposition 64 shows that *σ-generalisation* and *respect conflicts* are violated by Hoare_ρ for any argument-ranking semantics with respect to $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$, because the (usually) conflicting set A is among the most plausible sets. The highest-ranked argument in a set dominates all other arguments, implying that *composition* is satisfied if the underlying argument-ranking semantics satisfies *In*.

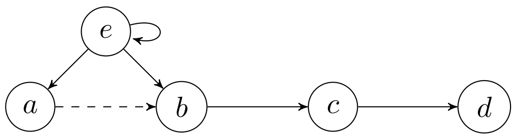

Figure 8.8: AF F_{25} from Example 97, where attack (a, b) is added later to obtain F'_{25}.

Proposition 65. *If ρ satisfies* In*, then* Hoare_ρ *satisfies* composition. *Proof: P. 195*

For *decomposition*, we can show that if ρ satisfies *VP*, *NaE*, and *In*, then *decomposition* is violated.

Proposition 66. *If ρ satisfies* VP*,* NaE*, and* In*, then* Hoare_ρ *violates* decomposition. *Proof: P. 195*

Next, we show that *weak reinstatement* is satisfied.

Proposition 67. Hoare_ρ *satisfies* weak reinstatement. *Proof: P. 195*

The proof of Proposition 67 shows that $E \cup \{a\} \sqsupset_F^{\mathsf{Hoare}_\rho} E$ cannot be guaranteed for every AF F, hence *strong reinstatement* is violated.

Adding attacks to an argument reduces its strength with respect to Cat, and in turn, increases the strength of the arguments attacked by the newly attacked argument. This behaviour leads to a violation of *addition robustness* by Hoare_{Cat}.

Example 97. *Let* $F_{25} = (\{a, b, c, d, e, f\}, \{(b, c), (c, d), (e, a), (e, b), (e, e)\})$ *be an AF, as depicted in Figure 8.8. Consider the two sets* $\{a, d\}$ *and* $\{b, c\}$*, we have:*

$$\{a, d\} \equiv_{F_{25}}^{\mathsf{Hoare}_{Cat}} \{b, c\}$$

since it holds that:

$$a \simeq_{F_{25}}^{Cat} b \simeq_{F_{25}}^{Cat} c \simeq_{F_{25}}^{Cat} d \simeq_{F_{25}}^{Cat} e \simeq_{F_{25}}^{Cat}$$

We can add attack (a, b) *to* F_{25}*, creating* $F'_{25} = (\{a, b, c, d, e, f\}, \{(b, c), (c, d), (e, a), (e, b), (e, e)\} \cup \{(a, b)\})$*. This addition changes the argument ranking to:*

$$c \succ_{F'_{25}}^{Cat} a \simeq_{F'_{25}}^{Cat} e \succ_{F'_{25}}^{Cat} d \succ_{F'_{25}}^{Cat} b$$

This implies:

$$\{b, c\} \sqsupset_{F_{25'}}^{\mathsf{Hoare}_{Cat}} \{a, d\}$$

Therefore, addition robustness *is violated.*

Example 97 can also illustrate the violation of *addition robustness* when using different argument-ranking semantics, such as the *Burden-based semantics* (Amgoud and Ben-Naim, 2013) or the argument-ranking semantics of Matt and Toni (2008).

Finally, we show that syntax independence is satisfied by Hoare_ρ.

Proposition 68. *If ρ satisfies* Abs*, then* Hoare_ρ *satisfies* syntax independence. *Proof: P. 195*

Since the *h-categoriser argument-ranking semantics* satisfies *Abs*, *In*, *VP*, and *NaE*, Hoare_{Cat} satisfies *composition* and *syntax independence*, and violates *decomposition*, as shown in Table 8.2.

Principles	Hoare_{Cat}
σ-generalisation	✗
respect conflicts	✗
composition	✓
decomposition	✗
weak reinstatement	✓
strong reinstatement	✗
addition robustness	✗
syntax independence	✓

Table 8.2: Principles satisfied and violated by Hoare_{Cat}.

8.3 GENERAL COMPARISON-BASED EXTENSION-RANKING SEMANTICS

The extension-ranking semantics previously discussed in this chapter focus on pair-wise comparisons between arguments. Another option is to evaluate the entire set of arguments based on the strength of all arguments contained. Before we define a new family of extension-ranking semantics, we need additional notations.

Gradual semantics assign each argument a numerical strength, which we can use to define a strength notion for a set of arguments that depends on the strength of each argument.

Definition 69. *Let $F = (A, R)$ be an AF and ρ a gradual semantics. We define the* numerical strength value *$\rho_F(E)$ for a set $E = \{a, b, \dots\} \subseteq A$ as a vector:*

$$\rho_F(E) = \langle \rho_F(a), \rho_F(b), \dots \rangle$$

The vector contains a value representing the strength of each argument in the set E. Next, we need to find ways of comparing these vectors. One approach is to aggregate the vectors and then compare the aggregated values.

Definition 70. *Let $V = \langle v_1, \dots, v_n \rangle$ and $V' = \langle v'_1, \dots, v'_m \rangle$ be two real-valued vectors. We call a relation between V and V' a* comparison relation $\square$, *if $\square$ induces a preorder.*

A variety of comparison relations can be found in the literature. In this work, we use five different ones: *summation*, *maxima*, *minima*, *lexicographical maxima*, *lexicographical minima*. A discussion of additional comparison relations can be found in (Dubois et al., 1996).

- *Summation* ($V \sqsupset^{\mathsf{sum}} V'$): The sum of all elements in V is bigger than the sum of elements in V', i. e, $\sum_{i=1}^{n} v_i > \sum_{j=1}^{m} v'_j$.
- *Maxima* ($V \sqsupset^{\mathsf{max}} V'$): The maximal element of V is bigger than the maximal element of V'.
- *Minima* ($V \sqsupset^{\mathsf{min}} V'$): The minimal element of V is bigger than the minimal element of V'.

- *Lexicographical Maxima* ($V \sqsupseteq^{\mathsf{leximax}} V'$): Rearranges the elements of the vectors in descending order, then there is an i such that $v_i > v'_i$ and for all $j < i$, $v_j = v'_j$, or for all i, $v_i = v'_i$ and $|V| > |V'|$. Additionally, $V =^{\mathsf{leximax}} V'$ iff for all i, $v_i = v'_i$.
- *Lexicographical Minima* ($V \sqsupseteq^{\mathsf{leximin}} V'$): Rearranges the elements of the vectors in ascending order, then there is an i such that $v_i > v'_i$ and for all $j < i$, $v_j = v'_j$, or for all i, $v_i = v'_i$ and $|V| > |V'|$. Additionally, $V =^{\mathsf{leximin}} V'$ iff for all i, $v_i = v'_i$.

Note that $\mathsf{max}(\emptyset) = \infty$ and $\mathsf{min}(\emptyset) = \infty$.

Example 98. *Consider the two vectors $v_1 = \langle 1, 2, 3, 4\rangle$ and $v_2 = \langle 1, 3, 5, 2\rangle$. If we compare these two vectors with $\square = \mathsf{sum}$, we get $\sum v_1 = 10$ and $\sum v_2 = 11$, thus $v_2 \geq^{\mathsf{sum}} v_1$. Using $\square = \mathsf{max}$, we get $\mathsf{max}(v_1) = 4$ and $\mathsf{max}(v_2) = 5$, so $v_2 \geq^{\mathsf{max}} v_1$. For $\square = \mathsf{min}$, we get $\mathsf{min}(v_1) = 1$ and $\mathsf{min}(v_2) = 1$, this means $v_1 =^{\mathsf{min}} v_2$. For $\square = \mathsf{leximax}$, we first have to arrange the two vectors in descending order: $v'_1 = \langle 4, 3, 2, 1\rangle$ and $v'_2 = \langle 5, 3, 2, 1\rangle$. Next, we compare the elements step by step. We begin with the first position, $4 < 5$, hence $v_2 \geq^{\mathsf{leximax}} v_1$. For $\square = \mathsf{leximin}$, we only need to order v_2, since v_1 is already in ascending order, so: $v'_2 = \langle 1, 2, 3, 5\rangle$. Again we begin with the first position $1 = 1$ and continue with the second and third positions ($2 = 2$ and $3 = 3$). At the fourth position, we get: $4 < 5$, hence $v_2 \geq^{\mathsf{leximin}} v_1$.*

A set with a better aggregated vector is more plausible to be accepted than other sets. With these comparison relations, we define a family of extension-ranking semantics. This family is a refinement of the *Order-based Extensions* by Konieczny et al. (2015), where only σ-extensions where considered.

Definition 71. *Let $F = (A, R)$ be an AF, a gradual semantics ρ, sets $E, E' \subseteq A$, and $\square$ an combination relation. We define the* general comparison-based extension-ranking semantics $\mathsf{GCB}_{\rho,\square}$ *via:*

$$E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\square}} E' \textit{ if and only if } \square(\rho_F(E)) \geq^{\square} \square(\rho_F(E'))$$

Note that $\sqsupseteq^{\mathsf{GCB}_{\rho,\square}}$ is a *total* preoder for $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.

Proposition 69. $\sqsupseteq^{\mathsf{GCB}_{\rho,\square}}$ *is a* total *preoder for* $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.

Proof: P. 196

Example 99. *Consider F_4 from Example 6 and sets $\{a\}$, $\{a, e, g\}$, $\{a, c, g\}$, and $\{a, d, g\}$. Using the* h-categoriser argument-ranking semantics*, the corresponding h-categoriser values are: $\mathsf{cat}(a) = 1$, $\mathsf{cat}(c) = 0.46$, $\mathsf{cat}(d) = 0.69$, $\mathsf{cat}(e) = 0.51$, and $\mathsf{cat}(g) = 0.74$. The corresponding vectors are:*

$$Cat(\{a\}) = \langle 1\rangle, \qquad Cat(\{a, e, g\}) = \langle 1, 0.51, 0.74\rangle,$$
$$Cat(\{a, c, g\}) = \langle 1, 0.46, 0.74\rangle, \qquad Cat(\{a, d, g\}) = \langle 1, 0.69, 0.74\rangle.$$

Using sum *as the underlying comparison relation results in the following ranking:*

$$\{a, d, g\} \sqsupset_{F_4}^{\mathsf{GCB}_{Cat,\mathsf{sum}}} \{a, e, g\} \sqsupset_{F_4}^{\mathsf{GCB}_{Cat,\mathsf{sum}}} \{a, c, g\} \sqsupset_{F_4}^{\mathsf{GCB}_{Cat,\mathsf{sum}}} \{a\}$$

Some argument-ranking semantics do not produce numerical values for each argument. One such example is the burden-based semantics (Amgoud and Ben-Naim, 2013), which only provides a preorder representing the relative strength of each argument. For instance, $a \succ_F^{Bbs} b$ means that a is stronger than b in F with respect to Bbs, but does not specify how much stronger a is compared to b. To address this, Bonzon et al. (2018) proposed a method to assign each argument a numerical strength value based on its position within a preorder. Note that we use a variation of the notion from Bonzon et al. (2018).

Definition 72. *Let $F = (A, R)$ be an AF and ρ be an argument-ranking semantics for AF. The* numerical strength value $sv_F^\rho(a) : A \to \mathbb{N}$ *of argument $a \in A$ in F with respect to ρ is defined as the length of the longest sequence of arguments $a_1, \dots, a_i$ such that $a_1 \succ_F^\rho \cdots \succ_F^\rho a_i \succ_F^\rho a$, then $sv_F^\rho(a) = \frac{|A|-i}{|A|}$. If there is no $b \in A$ such that $b \succ_F^\rho a$, then $sv_F^\rho(a) = 1$.*

An argument a gets the value 1 if it is the highest-ranked argument in an AF F with respect to argument-ranking semantics ρ. The arguments in the second position of the preorder receive the value $\frac{|A|-1}{|A|}$, and so on.

Example 100. *Consider F_4 from Example 6. Recall the induced argument ranking with respect to Bbs from Example 9:*

$$a \succ_{F_4}^{Bbs} g \succ_{F_4}^{Bbs} d \succ_{F_4}^{Bbs} b \succ_{F_4}^{Bbs} e \succ_{F_4}^{Bbs} c \succ_{F_4}^{Bbs} f$$

The numerical strength values $sv_{F_4}^{Bbs}$ are:

$$sv_{F_4}^{Bbs}(a) = 1 \qquad sv_{F_4}^{Bbs}(b) = \frac{4}{7} \qquad sv_{F_4}^{Bbs}(c) = \frac{2}{7}$$
$$sv_{F_4}^{Bbs}(d) = \frac{5}{7} \qquad sv_{F_4}^{Bbs}(e) = \frac{3}{7} \qquad sv_{F_4}^{Bbs}(f) = \frac{1}{7}$$
$$sv_{F_4}^{Bbs}(g) = \frac{6}{7}$$

With the notion sv, we define a corresponding gradual semantics for any argument-ranking semantics. While the gradual semantics already provide a numerical strength value for each argument, Definition 72 is still applicable. Fortunately, the resulting extension ranking $\mathsf{GCB}_{\rho,\Box}$ for gradual semantics ρ coincide for the combination relations $\Box \in \{\mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.

Proposition 70. *For an AF $F = (A, R)$, sets $E, E' \subseteq A$, and a gradual semantics ρ, it holds that for $\Box \in \{\mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$:*

$$E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\Box}} E' \text{ if and only if } E \sqsupseteq_F^{\mathsf{GCB}_{sv^\rho,\Box}} E'$$

Proof: P. 197

For $\square \in \{\mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$, we can generally assume that ρ is an argument-ranking semantics in Definition 71, to discuss the most general case. For $\square = \mathsf{sum}$, Proposition 70 does not hold in general.

Example 101. *Consider F_4 from Example 6 and the sets $\{a, c\}$ and $\{d, e\}$. As shown in Example 10, the corresponding h-categoriser ranking is:*

$$a \succ_{F_4}^{Cat} g \succ_{F_4}^{Cat} d \succ_{F_4}^{Cat} e \succ_{F_4}^{Cat} b \succ_{F_4}^{Cat} c \succ_{F_4}^{Cat} f$$

Using the numerical strength value $sv_{F_4}^{Cat}$, we get:

$$sv_{F_4}^{Cat}(\{a, c\}) = \langle \frac{1}{7}, \frac{6}{7} \rangle \quad \text{and} \quad sv_{F_4}^{Cat}(\{d, e\}) = \langle \frac{3}{7}, \frac{4}{7} \rangle$$

These two sets are equally plausible to be accepted when using $\mathsf{sum}(v)$ as comparision relation, i. e.:

$$\{a, c\} \equiv_{F_4}^{\mathsf{GCB}_{svCat,\mathsf{sum}}} \{d, e\}$$

The h-categoriser values are $Cat(a) = 1$, $Cat(c) = 0.46$, $Cat(d) = 0.69$, and $Cat(e) = 0.51$. So the aggregated values are:

$$\mathsf{sum}(\{1, 0.4\})) = 1.46 \quad \text{and} \quad \mathsf{sum}(\{0.69, 0.51\}) = 1.2$$

Therefore, we get:

$$\{a, c\} \sqsupset_{F_4}^{\mathsf{GCB}_{Cat,\mathsf{sum}}} \{d, e\}$$

Example 101 shows that we need to discuss $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ separately from $\mathsf{GCB}_{\rho,\square}$.

Next, we discuss the behaviour of $\mathsf{GCB}_{\rho,\square}$ in detail. We start with the observation that $\mathsf{GCB}_{\rho,\mathsf{max}}$ coincides with Hoare_ρ.

Proposition 71. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics, then $\sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{max}}} = \sqsupseteq_F^{\mathsf{Hoare}_\rho}$.* Proof: P. 197

Proposition 71 shows, that the satisfied and violated principles of $\mathsf{GCB}_{\rho,\mathsf{max}}$ can directly be inferred. Thus, $\mathsf{GCB}_{\rho,\mathsf{max}}$ satisfies *composition*, *weak reinstatement*, and *syntax independence*, and violates *σ-generalisation*, *respect conflicts*, *decomposition*, *strong reinstatement*, and *addition robustness* for $\sigma \in \{ad, co, gr, pr, st, sst\}$.

Next, we show that the set containing every argument is among the most plausible sets for $\square \in \{\mathsf{sum}, \mathsf{leximax}\}$.

Proposition 72. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics, then $A \in \mathtt{max}_{\mathsf{GCB}_{\rho,\square}}(F)$ for $\square \in \{\mathsf{sum}, \mathsf{leximax}\}$.* Proof: P. 197

Because of Proposition 72, we know that $\mathsf{GCB}_{\rho,\square}$ with $\square \in \{\mathsf{sum}, \mathsf{leximax}\}$ violates *σ-generalisation* and *respect conflicts* for all extension semantics discussed. For $\square \in \{\mathsf{min}, \mathsf{leximin}\}$, we also show that *$\sigma$-generalisation* is violated if ρ satisfies *VP*.

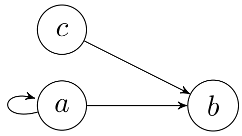

Figure 8.9: AF F_{26} from Example 103.

Example 102. *Consider F_5 from Example 22 and let ρ be an argument-ranking semantics satisfying* VP. *Then we know that $h \succ^{\rho}_{F_5} j$, so $\mathsf{min}(\rho_{F_5}(\{h, j\})) = \rho_{F_5}(j)$. Therefore, for $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$, we have:*

$$\{h\} \sqsupseteq^{\mathsf{GCB}_{\rho,\Box}}_{F_5} \{h, j\}$$

However, $\{h, j\}$ is admissible and also a complete, preferred, grounded, stable, and semi-stable extension for F_5, thus σ-generalisation *is violated for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{st}, \mathsf{sst}\}$.*

To show that *respect conflicts* is violated by $\mathsf{GCB}_{\rho,\Box}$ with $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$, we use the counterexample from Besnard et al. (2017). This example demonstrates the incompatibility between *CP* and *SC*.

Example 103. *Let $F_{26} = (\{a, b, c\}, \{(a, a), (a, b), (c, b)\})$ be an AF as depicted in Figure 8.9. If ρ satisfies* CP, *then $a \succ^{\rho}_{F_{26}} b$ and since $\mathsf{min}(\rho_{F_{26}}(\{a\})) > \mathsf{min}(\rho_{F_{26}}(\{b\}))$, we have for $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$:*

$$\{a\} \sqsupseteq^{\mathsf{GCB}_{\rho,\Box}}_{F_{26}} \{b\}$$

Hence, respect conflicts *is violated by $\mathsf{GCB}_{\rho,\Box}$ with $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$.*

Next, we show that $\mathsf{GCB}_{\rho,\Box}$ satisfies *composition* for all the comparison relations discussed so far. However, we first need to redefine *In* for gradual semantics, as suggested by Amgoud et al. (2017).

Definition 73. *A gradual semantics ρ satisfies* Independence for gradual semantics *if and only if for any two AFs $F = (A, R)$ and $F' = (A', R')$ such that $A \cap A' = \emptyset$ it holds that for all $a \in A$, $\rho_F(a) = \rho_{F \cup F'}(a)$.*

In other words, unconnected arguments should not affect the numerical strength value of an argument.

The following lemmas will help to show that $\mathsf{GCB}_{\rho,\Box}$ satisfies *composition* for gradual semantics ρ and $\Box \in \{\mathsf{leximax}, \mathsf{leximin}\}$, where for a vector of real numbers $V = \langle v_1, v_2, \dots \rangle$ and a number $c \in \mathbb{R}$, we define $c * V = \langle c * v_1, c * v_2, \dots \rangle$ as the scalar multiplication.

Lemma 4. *Let V, W, X, Y be vectors sorted in descending order such that $V \sqsupseteq^{\mathsf{leximax}} W$ and $X \sqsupseteq^{\mathsf{leximax}} Y$. It holds that $c_1 * V \cup c_2 * X \sqsupseteq^{\mathsf{leximax}} c_1 * W \cup c_2 * Y$ for $c_1, c_2 \in \mathbb{N}$.*

Proof: P. 198

Lemma 4 states a scalar multiplication as well as a union between two better ranked sets retains the relationship between vectors.

For $\mathsf{leximin}$, we can show the same behaviour.

Lemma 5. *Let V, W, X, Y be vectors sorted in ascending order such that $V \sqsupseteq^{\mathsf{leximin}} W$ and $X \sqsupseteq^{\mathsf{leximin}} Y$. It holds that $c_1 * V \cup c_2 * X \sqsupseteq^{\mathsf{leximin}} c_1 * W \cup c_2 * Y$ for $c_1, c_2 \in \mathbb{N}$.*
Proof: P. 200

Proposition 73. *If ρ satisfies* Independence for gradual semantics*, then* $\mathsf{GCB}_{\rho,\square}$ *satisfies* composition *for $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$.* *Proof: P. 200*

For the violation of *decomposition*, we can find a general result based on the principles that a gradual semantics ρ satisfies. Besides *Independence for gradual semantics*, we also need the principles *Equivalence* and *Maximality* (Amgoud et al., 2017).

Definition 74. *A gradual semantics ρ satisfies* Equivalence *if and only if for any AF $F = (A, R)$ and two arguments $a, b \in A$ there exits a bijective function f from a_F^- to b_F^- such that for all $x \in a_F^-$, $\rho_F(x) = \rho_F(f(x))$, then $\rho_F(a) = \rho_F(b)$.*

In other words, if the attackers of two arguments have the same strength, then the strength of those two arguments should be the same.

Definition 75. *A gradual semantics ρ satisfies* Maximality *if and only if for any AF $F = (A, R)$ and argument $a \in A$ such that $a_F^- = \emptyset$, then $\rho_F(a) = 1$.*

Informally, unattacked arguments should be given the highest possible strength value.

Proposition 74. *If ρ satisfies* Independence for gradual semantics*,* Equivalence*, and* Maximality*, then* $\mathsf{GCB}_{\rho,\square}$ *violates* decomposition *for $\square \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximax}\}$.*
Proof: P. 201

$\mathsf{GCB}_{\rho,\square}$ satisfies *weak reinstatement* for $\square = \mathsf{sum}$ and *strong reinstatement* for $\square = \mathsf{leximax}$.

Proposition 75. $\mathsf{GCB}_{\rho,\square}$ *satisfies* weak reinstatement *for $\square = \mathsf{sum}$ and* strong reinstatement *for $\square = \mathsf{leximax}$.* *Proof: P. 201*

For $\square \in \{\mathsf{min}, \mathsf{leximin}\}$, we show that *weak reinstatement* is violated for $\mathsf{GCB}_{\rho,\square}$ if ρ satisfies *Maximality*.

Example 104. *Consider F_5 from Example 22. If gradual semantics ρ satisfies* Maximality*, then $\rho_{F_5}(h) > \rho_{F_5}(j)$ and therefore $\mathsf{min}(\rho_{F_5}(\{h\})) > \mathsf{min}(\rho_{F_5}(\{h, j\}))$ Thus, for $\square \in \{\mathsf{min}, \mathsf{leximin}\}$:*

$$\{h\} \sqsupset_{F_5}^{\mathsf{GCB}_{\rho,\square}} \{h, j\}$$

Hence, weak reinstatement *is violated.*

To show that $\mathsf{GCB}_{Cat,\square}$ violates *addition robustness*, we need two counterexamples. We start with $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}\}$.

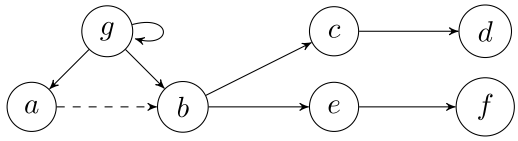

Figure 8.10: AF F_{27} from Example 105, where attack (a, b) is added later to obtain F'_{27}.

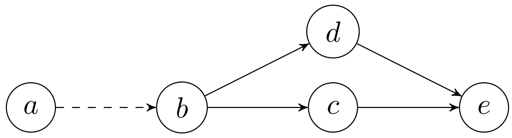

Figure 8.11: AF F_{28} from Example 106, where attack (a, b) is added later to obtain F'_{28}.

Example 105. *Consider $F_{27} = (\{a, b, c, d, e, f\}, \{(b, c), (b, e), (c, d), (e, f), (g, a), (g, b), (g, g)\})$ as depicted in Figure 8.10. If we use h-categoriser, then every argument receives the same strength value. Consider the sets $\{a, d, f\}$ and $\{b, c, e\}$. For combination relation $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}\}$, we get:*

$$\{a, d, f\} \equiv_{F_{27}}^{\mathsf{GCB}_{Cat,\Box}} \{b, c, e\}$$

We can add the attack (a, b) to obtain $F'_{27} = (\{a, b, c, d, e, f\}, \{(b, c), (b, e), (c, d), (e, f), (g, a), (g, b), (g, g)\} \cup \{(a, b)\})$. This addition causes the strength values of arguments c and e to increase, while the values of b, d, and f decrease. It follows that:

$$\{b, c, e\} \sqsupset_{F'_{27}}^{\mathsf{GCB}_{Cat,\Box}} \{a, d, f\}$$

for $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}\}$, thus demonstrating that addition robustness *is violated by* $\mathsf{GCB}_{Cat,\Box}$ *for* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}\}$.

Next, we examine $\mathsf{GCB}_{\rho,\Box}$ with $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$.

Example 106. *Consider $F_{28} = (\{a, b, c, d, e\}, \{(b, c), (b, d), (c, e), (d, e)\})$ as shown in Figure 8.11 and sets $\{a, e\}$ and $\{b, c\}$. For $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$, we have:*

$$\{a, e\} \equiv_{F_{28}}^{\mathsf{GCB}_{Cat,\Box}} \{b, c\}$$

We can add attack (a, b) to obtain $F'_{28} = (\{a, b, c, d, e\}, \{(b, c), (b, d), (c, e), (d, e)\} \cup \{(a, b)\})$. This results in :

$$\{b, c\} \sqsupset_{F'_{28}}^{\mathsf{GCB}_{Cat,\Box}} \{a, e\}$$

Therefore, addition robustness *is violated by* $\mathsf{GCB}_{Cat,\Box}$ *for* $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$.

To satisfy *syntax independence*, we must ensure that the underlying gradual semantics is not affected by the names of the arguments.

Proposition 76. *If gradual semantics ρ satisfies* Abs, *then* $\mathsf{GCB}_{\rho,\Box}$ *satisfies* syntax independence *for* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$. *Proof: P. 201*

In Example 101, we discussed the need to investigate $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ separately.

We start our principle investigation by showing that the set containing all arguments is one of the most plausible sets with respect to $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$.

Proposition 77. *Let $F = (A, R)$ be an AF, $A \in \mathtt{max}_{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}}(F)$ for argument-ranking-semantics ρ.* *Proof: P. 202*

The principles *σ-generalisation* and *respect conflicts* are violated by $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ for any argument-ranking semantics ρ.

Skiba (2023) has shown that sv^ρ can be influenced by unconnected and independent arguments, indicating that $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ violates *composition*.

Example 107. *Let $F_{29} = (\{a, b, c, d, e\}, \{(a, b), (b, a), (a, c), (b, c), (c, b), (c, c)\})$ be an AF as depicted in Figure 8.12, and let ρ be an argument-ranking semantics that satisfies* CP *and* Abs. *These principles imply the following argument ranking:*

$$d \equiv^\rho_{F_{29}} e \succ^\rho_{F_{29}} a \succ^\rho_{F_{29}} b \succ^\rho_{F_{29}} c$$

Since d and e have no attackers, they are the highest-ranked argument, followed by a, which has one attacker. Argument b has two attackers, and c has three attackers. F_{29} can be split into two AFs: $F_{29,1} = (\{a, b, c\}, \{(a, b), (b, a), (a, c), (b, c), (c, b), (c, c)\})$ and $F_{29,2} = (\{d, e\}, \emptyset)$. For $F_{29,1}$, the ranking is:

$$a \succ^\rho_{F_{29,1}} b \succ^\rho_{F_{29,1}} c$$

For $F_{29,2}$, we have:

$$d \simeq^\rho_{F_{29,2}} e$$

The corresponding numerical strength values are:

$$sv^\rho_{F_{29}}(a) = \frac{3}{5} \qquad sv^\rho_{F_{29}}(b) = \frac{2}{5} \qquad sv^\rho_{F_{29}}(c) = \frac{1}{5}$$
$$sv^\rho_{F_{29}}(d) = 1 \qquad sv^\rho_{F_{29}}(e) = 1$$

$$sv^\rho_{F_{29,1}}(a) = 1 \qquad sv^\rho_{F_{29,1}}(b) = \frac{4}{5} \qquad sv^\rho_{F_{29,1}}(c) = \frac{3}{5}$$
$$sv^\rho_{F_{29,2}}(d) = 1 \qquad sv^\rho_{F_{29,2}}(e) = 1$$

Next, consider the sets $\{a, b, d\}$ and $\{c, d\}$. We have:

$$\{a, b\} \sqsupset^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}}_{F_{29,1}} \{c\} \quad \textit{and} \quad \{d\} \equiv^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}}_{F_{29,2}} \{d\}$$

However, we get:

$$\{c, d\} \sqsupset^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}}_{F_{29}} \{a, b, d\}$$

Therefore, composition *is violated by* $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$.

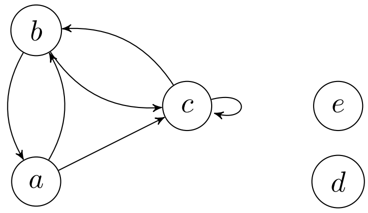

Figure 8.12: AF F_{29} for Example 107.

Example 107 shows that if an argument-ranking semantics satisfies *CP* and *Abs*, then $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ violates *composition*. Even though *h-categoriser argument-ranking semantics* violates *CP*, $\mathsf{GCB}_{sv^{Cat},\mathsf{sum}}$ still violates *composition* because Cat induces the same argument-ranking for F_{29}.

The violation of *composition* suggest that *decomposition* is also violated.

Example 108. *Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and ρ an argument-ranking semantics that satisfies* VP*. Let $a, c \in A_1$ and $b, d \in A_2$ with a and b are unattacked, so $a_F^- = b_F^- = \emptyset$, and c, d are attacked.* VP *implies $sv_F^\rho(a) > sv_F^\rho(c)$ and $sv_F^\rho(b) > sv_F^\rho(d)$. Consider the two sets $\{a, d\}$ and $\{b, c\}$, these two sets are comparable with $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$, so:*

$$\{a, d\} \sqsupseteq_F^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}} \{b, c\} \quad \textit{or} \quad \{b, c\} \sqsupseteq_F^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}} \{a, d\}$$

If we restrict ourselves to the subAFs, then VP *implies $sv_{F_1}^\rho(a) > sv_{F_1}^\rho(c)$ and $sv_{F_2}^\rho(b) > sv_{F_2}^\rho(d)$ and therefore:*

$$\{a\} \sqsupset_{F_1}^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}} \{c\} \quad \textit{and} \quad \{b\} \sqsupset_{F_2}^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}} \{d\}$$

Hence, if ρ satisfies VP*, then* decomposition *is violated.*

To show that *weak reinstatement* is satisfied by $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$, we can use a similar reasoning as for Proposition 75.

Proposition 78. $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ *satisfies* weak reinstatement *and violates* strong reinstatement. *Proof: P. 202*

$\mathsf{GCB}_{sv^{Cat},\mathsf{sum}}$ also violates *addition robustness*.

Example 109. *Consider again F_{27} from Example 105 and the sets $\{a, d, f\}$ and $\{b, c, e\}$. Again, all arguments are equal with respect to Cat, so we can add the attack between arguments a and b. The corresponding numerical strength scores for the modified AF are:*

$$sv_{F'_{27}}^{Cat}(a) = \frac{4}{6} \qquad sv_{F'_{27}}^{Cat}(b) = 0 \qquad sv_{F'_{27}}^{Cat}(c) = 1$$
$$sv_{F'_{27}}^{Cat}(d) = \frac{2}{6} \qquad sv_{F'_{27}}^{Cat}(e) = 1 \qquad sv_{F'_{27}}^{Cat}(f) = \frac{2}{6}$$

Principles	$\mathsf{GCB}_{Cat,\mathsf{sum}}$	$\mathsf{GCB}_{Cat,\mathsf{max}}$	$\mathsf{GCB}_{Cat,\mathsf{leximax}}$	$\mathsf{GCB}_{Cat,\mathsf{min}}$	$\mathsf{GCB}_{Cat,\mathsf{leximin}}$	$\mathsf{GCB}_{sv^{Cat},\mathsf{sum}}$
σ-generalisation	✗	✗	✗	✗	✗	✗
respect conflicts	✗	✗	✗	✗	✗	✗
composition	✓	✓	✓	✓	✓	✗
decomposition	✗	✗	✗	✗	✗	✗
weak reinstatement	✓	✓	✓	✗	✗	✓
strong reinstatement	✗	✗	✓	✗	✗	✗
addition robustness	✗	✗	✗	✗	✗	✗
syntax independence	✓	✓	✓	✓	✓	✓

Table 8.3: Principles satisfied and violated by $\mathsf{GCB}_{Cat,\square}$ for $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$, and $\mathsf{GCB}_{sv^{Cat},\mathsf{sum}}$ with $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

This results in:

$$\{b, c, e\} \sqsupset_{F'_{27}}^{\mathsf{GCB}_{sv^{Cat},\mathsf{sum}}} \{a, d, f\}$$

Therefore, $\mathsf{GCB}_{sv^{Cat},\mathsf{sum}}$ *violates* addition robustness.

$\mathsf{GCB}_{sv^{\rho},\mathsf{sum}}$ is only influenced by the names of the arguments if the underlying argument-ranking semantics is influenced by the names of the arguments. So if we are sure that ρ satisfies *Abs*, then $\mathsf{GCB}_{sv^{\rho},\mathsf{sum}}$ satisfies *syntax independence*.

Proposition 79. *If ρ satisfies* Abs *then* $\mathsf{GCB}_{sv^{\rho},\mathsf{sum}}$ *satisfies* syntax independence.

Proof: P. 202

The h-categoriser argument-rankings semantics satisfies *Abs*, *Independence for gradual semantics*, *Equivalence*, *Maximality*, and *VP*, and this semantics is total (Amgoud et al., 2017), therefore Table 8.3 shows the principles satisfied and violated by $\mathsf{GCB}_{Cat,\square}$ for $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$ and $\mathsf{GCB}_{sv^{Cat},\mathsf{sum}}$.

Connecting argument-ranking semantics with extension-based reasoning was already done by Konieczny et al. (2015). They propose to estimate the strength of an argument by counting the number of times it is part of an σ-extension.

Definition 76 (Konieczny et al. (2015))**.** *Let $F = (A, R)$ be an AF, σ an extension semantics, and $a \in A$. $ne_{\sigma,F}(a)$ is the normalised number of σ-extensions a is part of, i. e.,*

$$ne_{\sigma,F}(a) = \frac{|\{E \in \sigma(F) | a \in E\}|}{|\sigma(F)|}$$

The corresponding gradual semantics $ne_{\sigma,F}$ for two arguments $a, b \in A$ is:

$$a \succeq_F^{ne_\sigma} b \text{ if and only if } ne_{\sigma,F}(a) \geq ne_{\sigma,F}(b)$$

We can extend this definition to sets of arguments via:

$$ne_{\sigma,F}(E) = \{ne_{\sigma,F}(a), ne_{\sigma,F}(b), \dots\}$$

for a set of arguments $E = \{a, b, \dots\} \subseteq A$.

For any set of arguments, we can determine a vector containing, for each element, the number of times that argument is within a σ-extension.

Example 110. *Consider F_4 from Example 6. The corresponding complete extensions are $\{a\}$, $\{a,g\}$, $\{a,c,g\}$, and $\{a,d,g\}$. Hence, the vectors for ne_{co,F_4} are:*

$$ne_{\mathsf{co},F_4}\{a\} = \langle 1 \rangle, \qquad ne_{\mathsf{co},F_4}\{a,g\} = \langle 1, \frac{3}{4} \rangle,$$
$$ne_{\mathsf{co},F_4}\{a,c,g\} = \langle 1, \frac{1}{4}, \frac{3}{4} \rangle, \qquad ne_{\mathsf{co},F_4}\{a,d,g\} = \langle 1, \frac{1}{4}, \frac{3}{4} \rangle$$

For a more detailed discussion on counting the number of extensions an argument is apart of, including an investigation of the computational complexity of this problem, refer to the works of Fichte et al. (2019, 2024).

Konieczny et al. (2015) did not discuss $ne_{\sigma,F}$ as a gradual semantics, but rather as a notion for comparing sets of arguments. In this thesis, we only use $ne_{\sigma,F}$ in an general comparison-based extension-ranking semantics and therefore do not define $ne_{\sigma,F}$ as a proper gradual semantics. We leave the formal definition of gradual semantics and an in-depth analysis of the properties of $ne_{\sigma,F}$ to future work.

Example 111. *Let us consider F_4 from Example 6 and the sets $\{a\}, \{a,e,g\}, \{a,c,g\}$, and $\{a,d,g\}$. The corresponding vectors with respect to the complete extension semantics are: $ne_{\mathsf{co},F_4}\{a\} = \langle 1 \rangle$, $ne_{\mathsf{co},F_4}\{a,e,g\} = \langle 1, 0, \frac{3}{4} \rangle$, $ne_{\mathsf{co},F_4}\{a,c,g\} = \langle 1, \frac{1}{4}, \frac{3}{4} \rangle$, and $ne_{\mathsf{co},F_4}\{a,d,g\} = \langle 1, \frac{1}{4}, \frac{3}{4} \rangle$. When we use* sum *as the underlying comparison relation, the resulting ranking is:*

$$\{a,c,g\} \equiv_{F_4}^{\mathsf{GCB}_{ne_{\mathsf{co}},\mathsf{sum}}} \{a,d,g\} \sqsupset_{F_4}^{\mathsf{GCB}_{ne_{\mathsf{co}},\mathsf{sum}}} \{a,e,g\} \sqsupset_{F_4}^{\mathsf{GCB}_{ne_{\mathsf{co}},\mathsf{sum}}} \{a\}$$

If we use max *as our comparison relation, these four sets are all equally plausible to be accepted because they all contain argument a, which is a skeptically accepted argument with respect to the complete extension semantics. Hence, the choice of the comparison relation is important.*

The resulting ranking also differs from Example 99, where $\{a,e,g\}$ is more plausible to be accepted than $\{a,c,g\}$. This shows that the choice of numerical evaluation function is crucial for the resulting ranking.

Example 112. *Consider F_5 from Example 22. We are utilising ne_{ad} and* sum *and the resulting extension ranking $\sqsupseteq_{F_5}^{\mathsf{GCB}_{ne_{\mathsf{ad}},\mathsf{sum}}}$ is depicted in Figure 8.13. While the admissible set $\{h,j\}$ is still among the most plausible sets, the other admissible sets $\{h\}$ and $\emptyset$ are not. Also, the set containing every argument $\{h,i,j\}$ is among the most plausible sets.*

With $ne_{\sigma,F}$ and the comparison relations $\mathsf{sum}(v)$, $\mathsf{max}(v)$, $\mathsf{min}(v)$, $\mathsf{leximax}(v)$, and $\mathsf{leximin}(v)$, we receive a refinement of the *Order-based Extensions* proposed by Konieczny et al. (2015).

Definition 77 (Konieczny et al. (2015))**.** *Let $F = (A,R)$ be an AF, σ an extension semantics, and $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. We define the* Order-based Extensions $\mathsf{OBE}_{\sigma,\square}(F)$ *as:*

$$\mathsf{OBE}_{\sigma,\square}(F) = argmax_{E \in \sigma(F)} \square(ne_{\sigma,F}(E))$$

$$\begin{tikzcd}
\emptyset \arrow[r, dash] \arrow[d] \arrow[dr] & \{i\} \arrow[d] \arrow[dl] \\
\{j\} \arrow[r, dash] \arrow[d] \arrow[dr] & \{i,j\} \arrow[d] \arrow[dl] \\
\{h\} \arrow[r, dash] \arrow[d] \arrow[dr] & \{h,i\} \arrow[d] \arrow[dl] \\
\{h,j\} \arrow[r, dash] & \{h,i,j\}
\end{tikzcd}$$

Figure 8.13: Lattice depicting $\sqsupseteq_{F_5}^{\mathsf{GCB}_{ne_{\mathsf{ad}},\mathsf{sum}}}$ from Example 112.

The Order-based extensions are the extensions with the best aggregated value with respect to ne_σ and the comparison relation $\Box$. The extension-ranking semantics $\mathsf{GCB}_{ne_\sigma,\Box}$ can compare not only extensions but the entire powerset $\mathcal{P}(A)$.

Next, we investigate the principles that $\mathsf{GCB}_{ne_\sigma,\Box}$ satisfies.

In Example 112, we have already seen that the set containing all arguments A is one of the most plausible sets for $\sqsupseteq_{F_{22}}^{\mathsf{GCB}_{ne_{\mathsf{ad}},\mathsf{sum}}}$. We use this observation to show the violation of σ-generalisation, we show that A is always among the most plausible sets for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

Proposition 80. *Let $F = (A, R)$ be an AF, then $A \in \mathtt{max}_{\mathsf{GCB}_{ne_\sigma,\Box}}(F)$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ and $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}\}$.* *Proof: P. 202*

For $\Box = \{\mathsf{min}, \mathsf{leximin}\}$, the set containing all arguments is not among the most plausible sets for $\mathsf{GCB}_{ne_\sigma,\Box}$, but σ-generalisation is still violated.

Example 113. *Consider F_5 from Example 22. The set $\{h, j\}$ is a complete, preferred, and stable extension and also the grounded extension. However, $\mathsf{min}(ne_{\sigma,F_5}(\{h\})) > \mathsf{min}(ne_{\sigma,F_5}(\{h, j\}))$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{sst}\}$ and therefore:*

$$\{h\} \sqsupset_{F_5}^{\mathsf{GCB}_{ne_\sigma,\Box}} \{h, j\}$$

for all $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ and $\Box \in \{\min, \mathsf{leximin}\}$. Thus, σ-generalisation is violated for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{sst}\}$.

For $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}\}$, Proposition 80 is sufficient to show that *respect conflicts* is violated. For $\Box = \{\mathsf{min}, \mathsf{leximin}\}$, we show that *respect conflicts* is also violated.

Example 114. *Consider F_5 from Example 22 and the sets $\{i\}$ and $\{i,j\}$. For $\sigma \in \{\mathsf{ad},\mathsf{co},\mathsf{pr},\mathsf{gr},\mathsf{sst}\}$, $\mathsf{min}(ne_{\sigma,F_5}(\{i\})) = \mathsf{min}(ne_{\sigma,F_5}(\{i,j\}))$ and this implies for all $\sigma \in \{\mathsf{ad},\mathsf{co},\mathsf{gr},\mathsf{pr},\mathsf{sst}\}$.:*

$$\{i\} \equiv_{F_5}^{\mathsf{GCB}_{ne_\sigma,\mathsf{min}}} \{i,j\}$$

For $\Box = \mathsf{leximin}$, we get:

$$\{i,j\} \sqsupset_{F_5}^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximin}}} \{i\}$$

Thus, respect conflicts *is violated.*

The next lemma is be used to show that $\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}$ satisfies *composition* for $\sigma \in \{\mathsf{ad},\mathsf{co},\mathsf{pr},\mathsf{st}\}$.

Lemma 6. *Let $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ be two AFs such that $A_1 \cap A_2 = \emptyset$ and $F = F_1 \cup F_2$. Then it holds that $|\{E \in \sigma(F) \mid a \in E\}| = |\{E \in \sigma(F_1) \mid a \in E\}| * |\sigma(F_2)| + |\{E \in \sigma(F_2) \mid a \in E\}| * |\sigma(F_1)|$ for every $a \in A_1 \cup A_2$ and $\sigma \in \{\mathsf{ad},\mathsf{co},\mathsf{gr},\mathsf{pr},\mathsf{st},\mathsf{sst}\}$.* *Proof: P. 203*

Using the previous lemma in combination with Lemma 4 and Lemma 5, we show that $\mathsf{GCB}_{ne_\sigma,\Box}$ satisfies *composition* for $\sigma \in \{\mathsf{ad},\mathsf{co},\mathsf{pr},\mathsf{st},\mathsf{sst}\}$ and $\Box \in \{\mathsf{sum},\mathsf{max},\mathsf{leximax},\mathsf{min},\mathsf{leximin}\}$.

Proposition 81. $\mathsf{GCB}_{ne_\sigma,\Box}$ *satisfies* composition *for $\sigma \in \{\mathsf{ad},\mathsf{co},\mathsf{pr},\mathsf{st},\mathsf{sst}\}$ and $\Box \in \{\mathsf{sum},\mathsf{max},\mathsf{leximax},\mathsf{min},\mathsf{leximin}\}$.* *Proof: P. 203*

For the extension semantics $\sigma \in \{\mathsf{ad},\mathsf{co},\mathsf{pr},\mathsf{st},\mathsf{sst}\}$ and the comparision relations $\Box \in \{\mathsf{sum},\mathsf{max},\mathsf{leximax},\mathsf{min},\mathsf{leximin}\}$, we show that *decomposition* is violated for $\mathsf{GCB}_{ne_\sigma,\Box}$.

Example 115. *Let F_{30} be the AF as depicted in Figure 8.14. Consider the sets $\{a,e\}$ and $\{c,f,g\}$. F_{30} can be partitioned into two disjoint AFs $F_{30,1} = (\{a,b,c\},\{(a,b),(b,a),(a,c),(b,c)\}$ and $F_{30,2} = (\{d,e,f,g\},\{(d,e),(e,f),(e,g),(f,g),(g,f)\}$. The corresponding vectors are: $ne_{\mathsf{co},F_{30}}(\{a,e\}) = \langle \frac{1}{3}, 0\rangle$, $ne_{\mathsf{co},F_{30,1}}(\{a\}) = \langle \frac{1}{9}\rangle$, $ne_{\mathsf{co},F_{30,2}}(\{e\}) = \langle 0\rangle$, and $ne_{\mathsf{co},F_{30}}(\{c,f,g\}) = \langle 0, \frac{1}{3}, \frac{1}{3}\rangle$, $ne_{\mathsf{co},F_{30,1}}(\{c\}) = \langle 0\rangle$, and $ne_{\mathsf{co},F_{30,2}}(\{f,g\}) = \langle \frac{1}{9}, \frac{1}{9}\rangle$.*

For $\Box = \mathsf{sum}$: *We have $\{c,f,g\} \sqsupset_{F_{30}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\mathsf{sum}}} \{a,e\}$, but $\{a\} \sqsupset_{F_{30,1}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\mathsf{sum}}} \{c\}$.*

For $\Box \in \{\mathsf{max},\mathsf{leximax}\}$: *We have $\{a,e\} \equiv_{F_{30}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\Box}} \{c,f,g\}$, but $\{a\} \sqsupset_{F_{30,1}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\Box}} \{c\}$ and $\{f,g\} \sqsupset_{F_{30,2}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\Box}} \{e\}$.*

For $\Box \in \{\mathsf{min},\mathsf{leximin}\}$: *We have $\{a,e\} \equiv_{F_{30}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\Box}} \{c,f,g\}$, but $\{a\} \sqsupset_{F_{30,1}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\Box}} \{c\}$ and $\{f,g\} \sqsupset_{F_{30,2}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\Box}} \{e\}$.*

For $\sigma = \mathsf{gr}$, we need a different counterexample to show that $\mathsf{GCB}_{ne_{\mathsf{gr}},\Box}$ violates *decomposition* for $\Box \in \{\mathsf{sum},\mathsf{max},\mathsf{leximax},\mathsf{min},\mathsf{leximin}\}$.

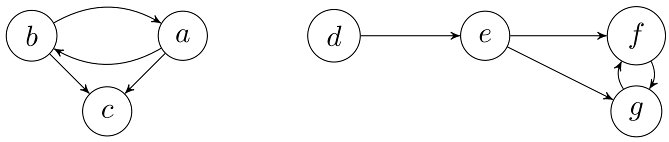

Figure 8.14: AF F_{30} from Example 115.

Figure 8.15: AF F_{31} from Example 116.

Example 116. *Let* $F_{31} = (\{a,b,c,d,e,f\}, \{(a,b),(b,c),(d,e),(e,f)\})$ *be an AF as depicted in Figure 8.15. Consider the sets* $\{a,c,e\}$ *and* $\{b,d,f\}$*. Then* F_{31} *can be partitioned into* $F_{31,1} = (\{a,b,c\}, \{(a,b),(b,c)\})$ *and* $F_{31,2} = (\{d,e,f\}, \{(d,e),(e,f)\})$*. For* $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$*, we have:*

$$\{a,c,e\} \equiv_{F_{31}}^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}} \{b,d,f\}$$

However, for $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$*, we have:*

$$\{a,c\} \sqsupset_{F_{31,1}}^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}} \{b} \quad \textit{and} \quad \{d,f\} \sqsupset_{F_{31,2}}^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}} \{e\}$$

Thus, decomposition *is violated for* $\mathsf{GCB}_{ne_{\mathsf{gr}},\square}$ *with* $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$.

Adding a defended argument to a set does not reduce the strength values of the other arguments in the set. Therefore, $\mathsf{GCB}_{ne_\sigma,\square}$ satisfies *weak reinstatement* for $\square \in \{\mathsf{sum}, \mathsf{max}\}$. For $\square = \mathsf{leximax}$, *strong reinstatement* is even satisfied.

Proposition 82. *For* $\square \in \{\mathsf{sum}, \mathsf{max}\}$ $\mathsf{GCB}_{ne_\sigma,\square}$ *satisfies* weak reinstatement*, and for* $\square = \mathsf{leximax}$ *satisfies* strong reinstatement *with* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof: P. 205

By adding an argument a to a set E, we can reduce the minima of E with respect to $ne_\sigma(E)$. Since argument a can be part of fewer σ-extensions than any other argument in E, *weak reinstatement* is violated by $\mathsf{GCB}_{ne_\sigma,\square}$ for $\square \in \{\mathsf{min}, \mathsf{leximin}\}$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Example 117. *Let* F_{32} *be an AF as depicted in Figure 8.16. Then every argument except* c *is part of three complete extensions. Argument* c *is defended by the set* $\{a,e\}$*, hence adding* c *into* $\{a,e\}$ *should not lower the plausibility of acceptance of this set. However,* $\mathsf{min}(ne_{\mathsf{co},F_{32}}(\{a,e\})) = \frac{1}{3}$ *but* $\mathsf{min}(ne_{\mathsf{co},F_{32}}(\{a,c,e\})) = \frac{1}{9}$*, for* $\square \in \{\mathsf{min}, \mathsf{leximin}\}$*, we get:*

$$\{a,e\} \sqsupset_{F_{32}}^{\mathsf{GCB}_{ne_{\mathsf{co}},\square}} \{a,c,e\}$$

Thus, weak reinstatement *is violated. The same behaviour can be observed for semantics* $\sigma \in \{\mathsf{ad}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

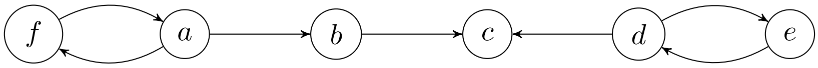

Figure 8.16: AF F_{32} from Example 117.

Only for $\sigma = \mathsf{gr}$, $\mathsf{GCB}_{ne_{\mathsf{gr}},\Box}$ for $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$ satisfies *weak reinstatement*.

Proposition 83. $\mathsf{GCB}_{ne_{\mathsf{gr}},\Box}$ *for* $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$ *satisfies* weak reinstatement.

Proof: P. 205

In the proof for Proposition 83, we show that $E \cup \{a\} \equiv_F^{\mathsf{GCB}_{ne_{\mathsf{gr}},\Box}} E$ for $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$, so *strong reinstatement* is violated.

To show that *addition robustness* is violated by $\mathsf{GCB}_{ne_\sigma,\Box}$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ and $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$, we need several counterexamples. We start with $\mathsf{GCB}_{ne_\sigma,\Box}$ such that $\sigma \in \{\mathsf{co}, \mathsf{pr}\}$ and $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$.

Example 118. *Let* F_{33} *be the AF depicted in Figure 8.17. Consider the sets* $\{a, e\}$ *and* $\{b, g\}$*. For* $\Box \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximax}\}$ *and* $\sigma \in \{\mathsf{co}, \mathsf{pr}\}$*, we have:*

$$\{a, e\} \equiv_{F_{33}}^{\mathsf{GCB}_{ne_\sigma,\Box}} \{b, g\}$$

Hence, we can add the attack (a, b) *to obtain* F'_{33}*. However, for this AF, we get:*

$$\{b, g\} \sqsupset_{F'_{33}}^{\mathsf{GCB}_{ne_\sigma,\Box}} \{a, e\}$$

For $\Box = \mathsf{max}$*, we consider the two sets* $\{a\}$ *and* $\{b, c\}$*. Again, we have:*

$$\{a\} \equiv_{F_{33}}^{\mathsf{GCB}_{ne_\sigma,max}} \{b, c\}$$

Therefore, we can add (a, b)*, but we get:*

$$\{b, c\} \sqsupset_{F'_{33}}^{\mathsf{GCB}_{ne_\sigma,max}} \{a\}$$

$\mathsf{GCB}_{ne_\sigma,\Box}$ *violates* addition robustness *for* $\sigma \in \{\mathsf{co}, \mathsf{pr}\}$ *and* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$.

Next, we examine $\mathsf{GCB}_{ne_{\sigma,},\Box}$ with $\sigma = \mathsf{ad}$ and $\Box \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$.

Example 119. *Let* F_{34} *be the AF depicted in Figure 8.18. Consider the sets* $\{a, d, e\}$ *and* $\{a, b, c\}$*. For* $\sigma = \mathsf{ad}$ *and* $\Box \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$*, we have:*

$$\{a, d, e\} \equiv_{F_{34}}^{\mathsf{GCB}_{ne_{\mathsf{ad}},\Box}} \{a, b, c\}$$

Thus, we get F'_{34} *by adding attack* (a, b)*. However, this results in:*

$$\{a, b, c\} \sqsupset_{F'_{34}}^{\mathsf{GCB}_{ne_{\mathsf{ad}},\Box}} \{a, d, e\}$$

So, addition robustness *is violated for* $\mathsf{GCB}_{ne_{\mathsf{ad}},\Box}$ *with* $\Box \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$.

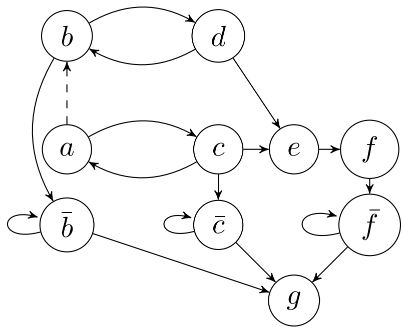

Figure 8.17: AF F_{33} from Example 118, where the dashed attack from a to b is added to obtain F'_{33}.

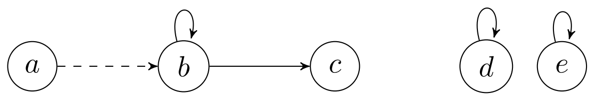

Figure 8.18: AF F_{34} from Example 119, where the dashed attack from a to b is added to obtain F'_{34}.

For $\mathsf{GCB}_{ne_\sigma,\Box}$ with $\sigma \in \{\mathsf{st}, \mathsf{sst}\}$ and $\Box \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$, we use the following example.

Example 120. *Let F_{35} be the AF depicted in Figure 8.19. Consider the sets $\{a, d, f\}$ and $\{b, c, e\}$, for $\sigma \in \{\mathsf{st}, \mathsf{sst}\}$ and $\Box \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$, we get:*

$$\{a, d, f\} \equiv_{F_{35}}^{\mathsf{GCB}_{ne_\sigma,\Box}} \{b, c, e\}$$

Hence, attack (a, b) can be added to obtain F'_{35}, but this results in:

$$\{b, c, e\} \sqsupset_{F'_{35}}^{\mathsf{GCB}_{ne_\sigma,\Box}} \{a, d, f\}$$

So, addition robustness *is violated for* $\mathsf{GCB}_{ne_\sigma,\Box}$ *with* $\Box \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$.

We continue with $\mathsf{GCB}_{ne_\sigma,\mathsf{max}}$ with $\sigma \in \{\mathsf{ad}, \mathsf{st}, \mathsf{sst}\}$.

Example 121. *Let F_{36} be the AF depicted in Figure 8.20. Consider the sets $\{a, e\}$ and $\{b, f\}$. For $\mathsf{GCB}_{ne_{\mathsf{ad}},\mathsf{max}}$, we get:*

$$\{a, e\} \equiv_{F_{36}}^{\mathsf{GCB}_{ne_{\mathsf{ad}},\mathsf{max}}} \{b, f\}$$

Hence, attack (a, b) can be added to obtain F'_{36}, but for this AF, we get:

$$\{b, f\} \sqsupset_{F'_{36}}^{\mathsf{GCB}_{ne_{\mathsf{ad}},\mathsf{max}}} \{a, e\}$$

For $\sigma \in \{\mathsf{st}, \mathsf{sst}\}$, we consider sets $\{a\}$ and $\{b, f\}$. We get:

$$\{a\} \equiv_{F_{36}}^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}} \{b, f\}$$

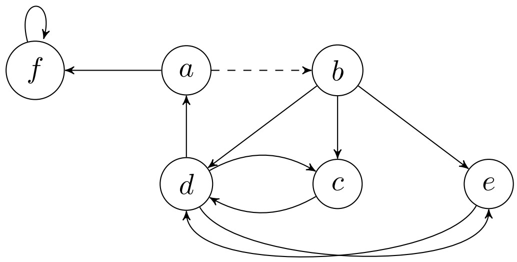

Figure 8.19: AF F_{35} from Example 120, where the dashed attack from a to b is added to obtain F'_{35}.

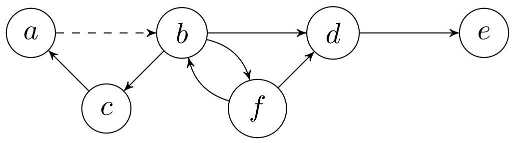

Figure 8.20: AF F_{36} from Example 121, where the dashed attack from a to b is added to obtain F'_{36}.

but we get:

$$\{b, f\} \sqsupset^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}}_{F_{36'}} \{a\}$$

So, addition robustness *is violated for* $\mathsf{GCB}_{ne_\sigma,\mathsf{max}}$ *with* $\sigma \in \{\mathsf{ad}, \mathsf{st}, \mathsf{sst}\}$.

Next, we discuss $\mathsf{GCB}_{ne_\sigma,\mathsf{min}}$ with $\sigma \in \{\mathsf{ad}, \mathsf{st}, \mathsf{sst}\}$.

Example 122. *Let* F_{37} *be the AF depicted in Figure 8.21. Consider the sets* $\{a, c\}$ *and* $\{b, c\}$. *For* $\sigma = \{\mathsf{ad}, \mathsf{st}, \mathsf{sst}\}$ *and* $\square = \mathsf{min}$, *we get:*

$$\{a, c\} \equiv^{\mathsf{GCB}_{ne_\sigma,\mathsf{min}}}_{F_{37}} \{b, c\}$$

Thus, attack (a, b) *can be added to obtain* F'_{37}*, however this results in:*

$$\{b, c\} \sqsupset^{\mathsf{GCB}_{ne_\sigma,\mathsf{min}}}_{F'_{37}} \{a, c\}$$

Therefore, addition robustness *is violated for* $\mathsf{GCB}_{ne_\sigma,min}$ *with* $\sigma = \{\mathsf{ad}, \mathsf{st}, \mathsf{sst}\}$.

We continue with the counterexamples for $\sigma = \mathsf{gr}$.

Example 123. *Let* F_{38} *be the AF as depicted in Figure 8.22. Consider the sets* $\{a, e, f\}$ *and* $\{b, c, d\}$. *For* $\sigma = \mathsf{gr}$ *and* $\square = \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$*, we get:*

$$\{a, e, f\} \equiv^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}}_{F_{38}} \{b, c, d\}$$

Hence, attack (a, b) *can be added to obtain* F'_{38}*, but this results in:*

$$\{b, c, d\} \sqsupset^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}}_{F'_{38}} \{a, e, f\}$$

So, addition robustness *is violated for* $\mathsf{GCB}_{ne_{\mathsf{gr}},\square}$ *with* $\square = \{\mathsf{sum}, \mathsf{leximax}, \mathsf{leximin}\}$.

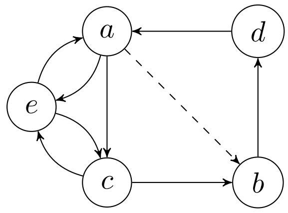

Figure 8.21: AF F_{37} from Example 122, where the dashed attack from a to b is added to obtain F'_{37}.

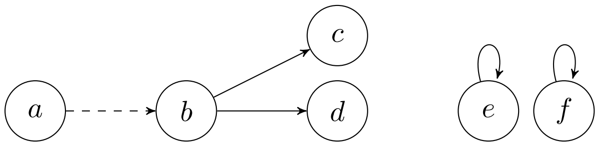

Figure 8.22: AF F_{38} from Example 123, where the dashed attack from a to b is added to obtain F'_{38}.

The final counterexample is for $\mathsf{GCB}_{ne_{\mathsf{gr}},\square}$ with $\square \in \{\mathsf{max}, \mathsf{min}\}$.

Example 124. *Let F_{39} be the AF depicted in Figure 8.23. Consider the sets $\{a, d\}$ and $\{b, e\}$. For $\sigma = \mathsf{gr}$ and $\square = \{\mathsf{max}, \mathsf{min}\}$, we get:*

$$\{a, d\} \equiv_{F_{39}}^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}} \{b, e\}$$

We obtain F'_{39} by adding attack (a, b), but this results in:

$$\{b, e\} \sqsupset_{F'_{39}}^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}} \{a, d\}$$

Therefore, addition robustness *is violated for* $\mathsf{GCB}_{ne_{\mathsf{gr}},\square}$ *with* $\square = \{\mathsf{max}, \mathsf{min}\}$.

Since ne_ρ does not relay on the names of the arguments, *syntax independence* is satisfied.

Proposition 84. $\mathsf{GCB}_{ne_\sigma,\square}$ *satisfies* syntax independence *for* $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. *Proof: P. 206*

The principles satisfies and violated by $\mathsf{GCB}_{ne_\sigma,\square}$ for $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$ and $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ are summarised in Table 8.4.

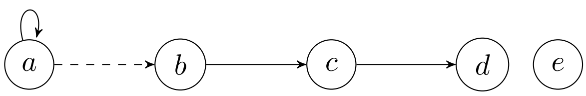

Figure 8.23: AF F_{39} from Example 124, where the dashed attack from a to b is added to obtain F'_{39}.

Principles	$\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}$	$\mathsf{GCB}_{ne_\sigma,\mathsf{max}}$	$\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}$	$\mathsf{GCB}_{ne_\sigma,\mathsf{min}}$	$\mathsf{GCB}_{ne_\sigma,\mathsf{leximin}}$
σ-generalisation	✗	✗	✗	✗	✗
respect conflicts	✗	✗	✗	✗	✗
composition	✓	✓	✓	✓	✓
decomposition	✗	✗	✗	✗	✗
weak reinstatement	✓	✓	✓	✗(✓$\sigma = \mathsf{gr}$)	✗(✓$\sigma = \mathsf{gr}$)
strong reinstatement	✗	✗	✓	✗	✗
addition robustness	✗	✗	✗	✗	✗
syntax independence	✓	✓	✓	✓	✓

Table 8.4: Principles satisfied and violated by $\mathsf{GCB}_{ne_\sigma,\Box}$ for $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$ and $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

8.4 TRADITIONAL LIFTING OPERATORS

The three families of extension-ranking semantics introduced in this chapter are all present in the formal argumentation literature (Bonzon et al., 2018; Amgoud and Ben-Naim, 2013; Konieczny et al., 2015). However, none of theses works use the term *lifting operators*. Technically, however, the extension rankings $\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_\rho})$, $\sqsupseteq_F^{\mathsf{Hoare}_\rho}$ and $\sqsupseteq_F^{\mathsf{GCB}_{\rho,\Box}}$ are all derived from the preorder $\succeq_F^\rho$ induced by the argument-ranking semantics ρ on abstract argumentation framework F, so they function as lifting operators. Lifting operators are discussed a few times in the context of formal argumentation, namely by Yun et al. (2018), Modgil and Prakken (2018) and Maly and Wallner (2021). The work of Yun et al. (2018) extends the ideas of Konieczny et al. (2015) and Bonzon et al. (2018), using lifting operators to compare σ-extensions rather than the powerset of arguments. Furthermore, the semantics proposed by Yun et al. (2018) are instances of the general comparison-based extension-ranking semantics, and are therefore already discussed in the previous section. Modgil and Prakken (2018) and Maly and Wallner (2021) discussed lifting operators in the context of structured argumentation. Modgil and Prakken (2018) used lifting operators as a way of incorporating preferences over defeasible rules in $ASPIC^+$. Maly and Wallner (2021) extend the discussion of Modgil and Prakken (2018) with an overview of other lifting operators that can be used for $ASPIC^+$. However, none of these works discuss lifting operators for abstract argumentation frameworks. In the following, we will discuss lifting operators in abstract argumentation in more detail.

Before we continue, we need to discuss the empty set separately. In the area of lifting operators, the empty set is usually disregarded because it does not fit into the motivation of finding the "best" committee. Unfortunately, this has the side effect that the lifting operators cannot handle the empty set well, or sometimes not at all. In Example 96, we saw that for Hoare's lifting operator, $\emptyset$ is the worst-ranked set of arguments. However, this set is always admissible, and for some abstract argumentation frameworks, $\emptyset$ is even the only complete extension. Therefore, $\emptyset$ should not be the worst-ranked set. In the context of *ASPIC*$^+$, Modgil and Prakken (2018) have argued that arguments without defeasible rules are strictly stronger than arguments with defeasible rules. So, the empty set is stronger than any other set. We can translate this as:

The empty set is strictly more plausible to be accepted than any non-empty set of arguments.

One problem with this statement is that the empty set is the only most plausible set in any abstract argumentation framework, implying that σ*-generalisation* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ cannot be satisfied. However, treating the empty set differently without losing transitivity or expressivity is challenging.

Example 125. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ be two sets of arguments such that for an extension-ranking semantics τ:*

$$E \sqsupset^{\tau}_{F} E'$$

If we assume that $\emptyset$ is equally plausible to be accepted as every other set of arguments with respect to τ in F, then we have:

$$E \equiv^{\tau}_{F} \emptyset \quad \text{and} \quad E' \equiv^{\tau}_{F} \emptyset$$

Since τ induces a preorder, we know that $\sqsupseteq^{\tau}_{F}$ must be transitive, implying:

$$E \equiv^{\tau}_{F} E'$$

Thus, the entire preoder collapses to a single layer or is not even a preorder anymore. Weakening the order to a non-strict one leads to the same behaviour.

*The only option left would be to make $\emptyset$ incomparable to any other set. However, this implies that $\emptyset$ is always part of the most plausible sets, and therefore σ-*generalisation *for $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ is violated.*

Example 125 shows that there is no good way to handle the empty set in general. We adopt the sentiments of Modgil and Prakken (2018) for our setting. Later, we will see that this choice has no effect on satisfying the principles of extension-ranking semantics discussed.

In the rest of this chapter, we discuss three lifting operators and propose the corresponding extension-ranking semantics.

8.4.1 Kelly's Extension

We start with the most restrictive operator, called *Kelly's Extension*. This operator is well-known in the field of computational social choice and is particularly useful for studying strategy proofness (Barberà, 2011; Brandt et al., 2016a). Kelly's Extension states that if all elements of a set E' are ranked below all elements of a set E, then E is more plausible to be accepted than E'.

Definition 78. *Let $F = (A, R)$ be an AF, ρ an argument-ranking semantics, and $E, E' \subseteq A$. We define* Kelly's extension-ranking semantics Kelly_{ρ} *via:*

1. *If* $E = E'$ *then* $E \equiv_F^{\mathsf{Kelly}_\rho} E'$*;*

2. *If* $E = \emptyset$ *and* $E' \neq \emptyset$ *then* $E \sqsupset_F^{\mathsf{Kelly}_\rho} E'$*;*

3. *Otherwise* $E \sqsupseteq_F^{\mathsf{Kelly}_\rho} E'$ *if and only if* $a \succ_F^\rho b \; \forall a \in E$ *and* $\forall b \in E'$.

In other words, a set of arguments E is at least as plausible to be accepted as E' in an abstract argumentation framework F if every argument of E is at least as strong as every argument of E' in F with respect to an argument-ranking semantics ρ. Note that $\emptyset$ must be treated separately, as discussed earlier.

Proposition 85. Kelly_ρ *is transitive and reflexive.* *Proof: P. 206*

Example 126. *Consider AF* F_4 *from Example 6. The argument ranking according to* Cat *is:*

$$a \succ_{F_4}^{Cat} g \succ_{F_4}^{Cat} d \succ_{F_4}^{Cat} e \succ_{F_4}^{Cat} b \succ_{F_4}^{Cat} c \succ_{F_4}^{Cat} f$$

Consider the two sets $\{a, d, g\}$ *and* $\{b, e\}$*. Every element of the first set is ranked above every element of the second set according to* Cat*, hence:*

$$\{a, d, g\} \sqsupset_{F_4}^{\mathsf{Kelly}_{Cat}} \{b, e\}$$

So, in this case the complete set $\{a, d, g\}$ *is more plausible to be accepted than* $\{b, e\}$*, which is our intended behaviour. Now consider the two complete sets* $\{a, d, g\}$ *and* $\{a, c, g\}$*. These two sets are incomparable according to Kelly's extension-ranking semantics.*

Example 127. *Consider AF* F_5 *from Example 22. The argument ranking induced by* Cat *is:*

$$h \succ_{F_5}^{Cat} j \succ_{F_5}^{Cat} i$$

For the two admissible sets $\{h\}$ *and* $\{h, j\}$*, we get :*

$$\{h, i\} \sqsupset_{F_5}^{\mathsf{Kelly}_{Cat}} \{j\} \quad \text{and} \quad \{h\} \sqsupset_{F_5}^{\mathsf{Kelly}_{Cat}} \{i, j\}$$

However, we have:

$$\{h, i\} \asymp_{F_5}^{\mathsf{Kelly}_{Cat}} \{h\}$$

So, the two admissible sets are incomparable to each other, but there is no set (except $\emptyset$*) that is strictly better than these two sets.*

For Kelly's extension-ranking semantics*, we see that a number of sets are incomparable. In this example, the sets* $\{h, i\}$ *and* $\{h, i, j\}$ *are incomparable to every other set except* $\emptyset$*.*

The full extension ranking is shown in Figure 8.24.

Example 127 shows that *Kelly's extension-ranking semantics* has several incompatibilities. Specifically, two sets which are not disjoint are incomparable. Even if we treat $\emptyset$ separately, Kelly_ρ still violates *σ-generalisation*.

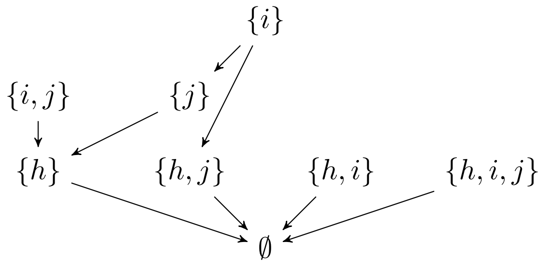

Figure 8.24: $\sqsupset_{F_5}^{\mathsf{Kelly}_{Cat}}$ from Example 127 depicted as a lattice.

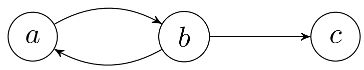

Figure 8.25: AF F_{40} from Example 128.

Example 128. *Let* $F_{40} = (\{a, b, c\}, \{(a, b), (b, a), (b, c)\})$ *be an AF as depicted in Figure 8.25. The non-empty admissible sets are* $\{a\}$*,* $\{a, c\}$ *and* $\{b\}$*.*

First, consider an argument-ranking semantics ρ_1*, where* $a \succ_{F_{40}}^{\rho_1} b$*. Then:*

$$\{a\} \sqsupset_{F_{40}}^{\mathsf{Kelly}_{\rho_1}} \{b\}$$

Thus, $\{b\} \notin \mathtt{max}_{\mathsf{Kelly}_{\rho_1}}(F_{40})$*.*

Next, assume an argument-ranking semantics ρ_2 *with* $b \succ_{F_{40}}^{\rho_2} a$*. This implies:*

$$\{b\} \sqsupset_{F_{40}}^{\mathsf{Kelly}_{\rho_2}} \{a\}$$

It must hold that a *is equally strong as* b*.*

Consider argument-ranking semantics ρ_3 *such that* $a \simeq_{F_{40}}^{\rho_3} b \succ_{F_{40}}^{\rho_3} c$*. Then there is no set (except* $\emptyset$*) that is strictly more plausible to be accepted than* $\{a, b\}$*. So, if we treat* $\emptyset$ *differently,* $\{a, b\}$ *would be among the most plausible sets in* F_{40} *with respect to* ρ_3*.*

Let ρ_4 *be an argument-ranking semantics with* $c \succ_{F_{40}}^{\rho_4} a \simeq_{F_{40}}^{\rho_4} b$*. Then:*

$$\{c\} \sqsupset_{F_{40}}^{\mathsf{Kelly}_{\rho_4}} \{a\}$$

The last case we examine is an argument ranking where all arguments are equally strong. Consider argument-ranking semantics ρ_5 *such that* $a \simeq_{F_{40}}^{\rho_5} b \simeq_{F_{40}}^{\rho_5} c$*. Then every non-empty set of arguments is incomparable to each other, implying that either only* $\emptyset$ *is the only most plausible set, or every set is among the most plausible sets.*

Example 128 shows that there is no possible argument ranking that satisfies the condition for *σ-generalisation* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$, and also that the choice of how to handle $\emptyset$ is not relevant to violating *σ-generalisation*.

Example 128 also shows that *respect conflicts* is violated. Consider ρ_3 again, then:

$$\{a, b\} \sqsupset_{F_{40}}^{\mathsf{Kelly}_{\rho_3}} \{c\}$$

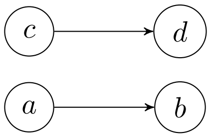

Figure 8.26: Recalled AF F_8 from Example 29.

Kelly_ρ also violates *composition* for any *universally realisable* argument-ranking semantics ρ.

Example 129. *Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and ρ an argument-ranking semantics that satisfies* UR *and* In.

Assume $a, b \in A_1$ and $c, d \in A_2$ with (partial) argument rankings:

$$a \succ_F^\rho b \succ_F^\rho c \succ_F^\rho d$$

Consider sets $\{a, c\}$ and $\{b, d\}$, because of In*:*

$$\{a\} \sqsupseteq_{F_1}^{\mathsf{Kelly}_\rho} \{b\} \quad \text{and} \quad \{c\} \sqsupseteq_{F_2}^{\mathsf{Kelly}_\rho} \{d\}$$

However, $c \not\succ_F^\rho b$ and therefore:

$$\{a, c\} \asymp_F^{\mathsf{Kelly}_\rho} \{b, d\}$$

Thus, if ρ satisfies UR *and* In*, then Kelly_ρ violates* composition.

If ρ satisfies *VP*, then Kelly_ρ violates *decomposition*.

Example 130. *Consider F_8 from Example 29 (recalled in Figure 8.26) and sets $\{a, c\}$ and $\{b\}$. If argument-ranking semantics ρ satisfies* VP*, then $a \succ_{F_8}^\rho b$ and $c \succ_{F_8}^\rho b$, hence:*

$$\{a, c\} \sqsupset_{F_8}^{\mathsf{Kelly}_\rho} \{b\}$$

Next, consider the subAF $F_{8,2} = (\{c, d\}, \{(c, d)\})$. Based on the definition of Kelly's extension-ranking semantics, we have:

$$\emptyset \sqsupset_{F_{8,2}}^{\mathsf{Kelly}_\rho} \{c\}$$

Therefore, if ρ satisfies VP*, then Kelly_ρ violates* decomposition.

To avoid violating *decomposition*, we would have to treat the empty set differently, but as already discussed in Example 125, we cannot weaken the relation to a non-strict order, as this will flatten the entire ranking. If the empty set is incomparable or equally plausible to be accepted to any other set, then the same reasoning as in Example 130 still applies.

The fact that non-disjoint sets of arguments are incomparable to each other, implies that *weak reinstatement* is violated by Kelly_ρ.

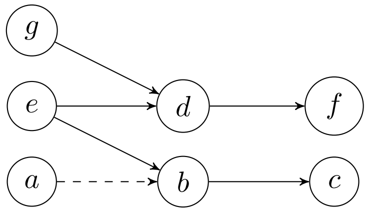

Figure 8.27: AF F_{41} from Example 131, where attack (a, b) is added later to obtain $F_{41'}$.

Proposition 86. Kelly_ρ *violates* weak reinstatement *for every argument-ranking semantics* ρ. *Proof: P. 206*

Kelly_{Cat} violates *addition robustness*.

Example 131. *Consider AF* F_{41} *as depicted in Figure 8.27. The argument ranking induced by* Cat *is:*

$$a \simeq^{Cat}_{F_{41}} e \simeq^{Cat}_{F_{41}} g \succ^{Cat}_{F_{41}} f \succ^{Cat}_{F_{41}} c \succ^{Cat}_{F_{41}} b \succ^{Cat}_{F_{41}} d$$

Consider the sets $\{a, f\}$ *and* $\{b, c\}$*. Then:*

$$\{a, f\} \sqsupset^{\mathsf{Kelly}_{Cat}}_{F_{41}} \{b, c\}$$

We add the attack between a *and* b *to get* F'_{41}*. However, this addition changes the induced argument ranking to*

$$a \simeq^{Cat}_{F'_{41}} e \simeq^{Cat}_{F'_{41}} g \succ^{Cat}_{F'_{41}} f \simeq^{Cat}_{F'_{41}} c \succ^{Cat}_{F'_{41}} b \simeq^{Cat}_{F'_{41}} d$$

This change in the argument ranking also causes a change in the extension ranking:

$$\{a, f\} \asymp^{\mathsf{Kelly}_{Cat}}_{F'_{41}} \{b, c\}$$

Thus, $\{a, f\}$ *is no longer at least as plausible to be accepted as* $\{b, c\}$*, and therefore* addition robustness *is violated.*

Finally, Kelly_ρ satisfies *syntax independence* as long as the underlying argument-ranking semantics satisfies *Abs*.

Proposition 87. *If* ρ *satisfies* Abs *then* Kelly_ρ *satisfies* syntax independence. *Proof: P. 206*

The *h-categoriser argument-ranking semantics* satisfies *Abs*, *In*, *VP* and *UR*, so Kelly_{Cat} satisfies *syntax independence* as shown in Table 8.5.

Principles	Kelly_{Cat}
σ-generalisation	✗
respect conflicts	✗
composition	✗
decomposition	✗
weak reinstatement	✗
strong reinstatement	✗
addition robustness	✗
syntax independence	✓

Table 8.5: Principles satisfied and violated by Kelly_{Cat}.

8.4.2 Fishburn

One of the disadvantages of Kelly's extension-ranking semantics is that two non-disjoint sets of arguments are always incomparable. Such behaviour is not intended for abstract argumentation. Let us recall the reinstatement principle, were an argument a that is defended by a set E and does not create new conflict should be included in E. However, for Kelly's extension-ranking semantics the sets E and $E \cup \{a\}$ are incomparable.

A relaxation of Kelly's extension is known as *Fishburn's Extension*, where a set E is at least as plausible to be accepted as E' if all elements that are only in E are ranked better than all elements in E', and all elements that are exclusive in E' are ranked worse than all elements of E.

Definition 79. *Let $F = (A, R)$ be an AF, ρ an argument-ranking semantics, and $E, E' \subseteq A$. We define* Fishburn's extension-ranking semantics $\mathsf{Fishburn}_\rho$ *via:*

1. *If $E = E'$, then $E \equiv_F^{\mathsf{Fishburn}_\rho} E'$;*
2. *If $E = \emptyset$ and $E' \neq \emptyset$, then $E \sqsupset_F^{\mathsf{Fishburn}_\rho} E'$;*
3. *Otherwise, $E \sqsupseteq_F^{\mathsf{Fishburn}_\rho} E'$ if and only if $a \succ_F^\rho b$, $a \succ_F^\rho x, x \succ_F^\rho b$ $\forall a \in E \setminus E', \forall b \in E' \setminus E$ and $x \in E \cap E'$*

Proposition 88. $\mathsf{Fishburn}_\rho$ *is reflexive and transitive.* *Proof: P. 206*

Example 132. *Consider AF F_4 from Example 6. The argument ranking according to Cat is:*

$$a \succ_{F_4}^{Cat} g \succ_{F_4}^{Cat} d \succ_{F_4}^{Cat} e \succ_{F_4}^{Cat} b \succ_{F_4}^{Cat} c \succ_{F_4}^{Cat} f$$

Consider the sets $\{b, d, g\}$ and $\{a, d, g\}$. First, we compare the exclusive arguments of each of these two sets, b and a (where $a \succ_{F_4}^{Cat} b$). Next, we need to show that a is stronger than every argument these two sets have in common, d and g (where $a \succ_{F_4}^{Cat} d$ and $a \succ_{F_4}^{Cat} g$). The last step is to show that both d and g are stronger than b. Therefore:

$$\{a, d, g\} \sqsupset_{F_4}^{\mathsf{Fishburn}_{Cat}} \{b, d, g\}$$

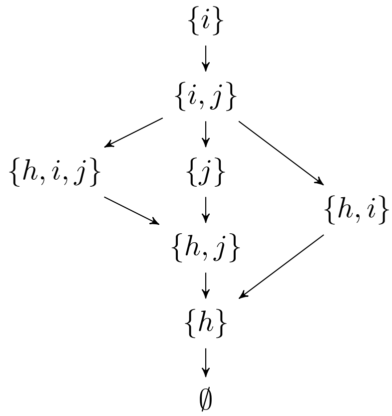

Figure 8.28: $\sqsupseteq_{F_5}^{\mathsf{Fishburn}_{Cat}}$ from Example 133 depicted as a lattice.

Example 133. *Consider AF F_5 from Example 22 and the* h-categoriser argument-ranking semantics. *The set $\{h\}$ is strictly more plausible to be accepted than any non-empty set of arguments, since h is the best ranked argument according to Cat. The extension ranking $\sqsupseteq_{F_5}^{\mathsf{Fishburn}_{Cat}}$ is depicted in Figure 8.28.*

We would like to point out some interesting observations in this extension ranking.

- *The admissible set $\{h, j\}$ is strictly less plausible to be accepted than $\{h\}$, thus* σ-generalisation *is violated for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.*

- $\{h\} \sqsupset_{F_5}^{\mathsf{Fishburn}_{Cat}} \{h, j\}$ *also shows that* weak reinstatement *is violated, because argument h defends j.*

- $\{i, j\} \sqsupset_{F_5}^{\mathsf{Fishburn}_{Cat}} \{i\}$*, so the conflicting set $\{i, j\}$ is ranked better than the conflict-free set $\{i\}$, implying that* respect conflicts *is violated.*

Example 133 already shows a number of violations of principles, even though the abstract arugmentation framework is quite simple. Other argument-ranking semantics, such as *burden-based argument-ranking semantics* or *serialisability argument-ranking semantics*, induce the same ranking. Thus, $\mathsf{Fishburn}_\rho$ also violates the same principles with respect to these argument-ranking semantics ρ.

Based on the results of Example 133, one might assume that it is possible to satisfy ad-*soundness*, but this is not true.

Example 134. *Assume AF* $F_{42} = (\{a, b, c, d\}, \{(a, b), (b, a), (a, c), (b, c), (c, d)\})$ *as depicted in Figure 8.29 and the* h-categoriser argument-ranking semantics*:*

$$d \succ_{F_{42}}^{Cat} a \simeq_{F_{42}}^{Cat} b \succ_{F_{42}}^{Cat} c$$

$\{d\}$ *is strictly more plausible to be accepted than the admissible set $\{a, d\}$ according to* $\mathsf{Fishburn}_{Cat}$*, but $\{d\}$ is not admissible. Hence,* ad-soundness *violated.*

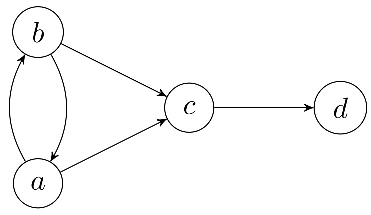

Figure 8.29: AF F_{42} from Example 134.

To show that *composition* is violated, we use the same reasoning as for Kelly_ρ in Example 129.

Example 135. *Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and ρ an argument-ranking semantics that satisfies* UR *and* In*. Assume $a, b \in A_1$ and $c, d \in A_2$ with argument rankings:*

$$a \succ^\rho_F b \succ^\rho_F c \succ^\rho_F d$$

Consider sets $\{a, c\}$ and $\{b, d\}$, then because of In*:*

$$\{a\} \sqsupseteq^{\mathsf{Fishburn}_\rho}_{F_1} \{b\} \quad \textit{and} \quad \{c\} \sqsupseteq^{\mathsf{Fishburn}_\rho}_{F_2} \{d\}$$

However, $c \not\succ^\rho_F b$ and therefore:

$$\{a, c\} \asymp^{\mathsf{Fishburn}_\rho}_F \{b, d\}$$

Thus, if ρ satisfies UR *and* In*, then* $\mathsf{Fishburn}_\rho$ *violates* composition.

To show that $\mathsf{Fishburn}_\rho$ violates *decomposition*, we use the same reasoning as for Kelly_ρ. So, if ρ satisfies *VP*, then *decomposition* is violated.

Example 136. *Consider F_8 from Example 29 and sets $\{a, c\}$ and $\{b\}$. If an argument-ranking semantics satisfies* VP*, then $a \succ^\rho_{F_8} b$ and $c \succ^\rho_{F_8} b$. Hence:*

$$\{a, c\} \sqsupset^{\mathsf{Fishburn}_\rho}_{F_8} \{b\}$$

But it holds that:

$$\emptyset \sqsupset^{\mathsf{Fishburn}_\rho}_{F_{8,2}} \{c\}$$

Thus, decomposition *is violated by* $\mathsf{Fishburn}_\rho$*, if ρ satisfies* VP.

To show that *addition robustness* is violated by $\mathsf{Fishburn}_{Cat}$, we use Example 131 again.

Example 137. *Consider F_{41} from Example 131. The argument ranking induced by Cat is:*

$$a \simeq^{Cat}_{F_{41}} e \simeq^{Cat}_{F_{41}} g \succ^{Cat}_{F_{41}} f \succ^{Cat}_{F_{41}} c \succ^{Cat}_{F_{41}} b \succ^{Cat}_{F_{41}} d$$

Principles	$\mathsf{Fishburn}_{Cat}$
σ-generalisation	✗
respect conflicts	✗
composition	✗
decomposition	✗
weak reinstatement	✗
strong reinstatement	✗
addition robustness	✗
syntax independence	✓

Table 8.6: Principles satisfied and violated by $\mathsf{Fishburn}_{Cat}$.

We consider sets $\{a, f\}$ and $\{b, c\}$, for which we have:

$$\{a, f\} \sqsupset_{F_{41}}^{\mathsf{Fishburn}_{Cat}} \{b, c\}$$

Then we add an attack from a to b to get $F_{41'}$ and change the argument ranking accordingly to:

$$a \simeq_{F'_{41}}^{Cat} e \simeq_{F'_{41}}^{Cat} g \succ_{F'_{41}}^{Cat} f \simeq_{F'_{41}}^{Cat} c \succ_{F'_{41}}^{Cat} b \simeq_{F'_{41}}^{Cat} d$$

Argument f is no longer strictly stronger than c, so we get:

$$\{a, f\} \asymp_{F_{41}}^{\mathsf{Fishburn}_{Cat}} \{b, c\}$$

Therefore, addition robustness *is violated.*

$\mathsf{Fishburn}_\rho$ satisfies *syntax independence* if ρ satisfies *Abs*.

Proposition 89. *If ρ satisfies* Abs*, then* $\mathsf{Fishburn}_\rho$ *satisfies* syntax independence.

Proof: P. 207

In summary, $\mathsf{Fishburn}_{Cat}$ satisfies only *syntax independence*, because Cat satisfies *UR*, *VP*, and *In* as depicted in Table 8.6.

8.4.3 Elitist

Another relaxation of Kelly's extension is the *Elitist operator*, introduced by Modgil and Prakken (2018) in the context of structured argumentation. Whereas Fishburn's extension weakens Kelly's extension by allowing overlapping elements, for the Elitist operator the weakest element must be beaten by every argument.

Definition 80. *Let $F = (A, R)$ be an AF, ρ an argument-ranking semantics and $E, E' \subseteq A$. We define* Elitist extension-ranking semantics Eli_ρ *via:*

1. *If $E = E'$ then $E \equiv_F^{\mathsf{Eli}_\rho} E'$;*
2. *If $E = \emptyset$ and $E' \neq \emptyset$ then $E \sqsupset_F^{\mathsf{Eli}_\rho} E'$;*

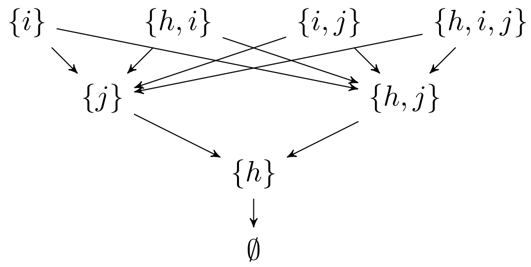

Figure 8.30: $\sqsupseteq_{F_5}^{\mathsf{Eli}_{Cat}}$ from Example 139 depicted as a lattice.

3. *Otherwise* $E \sqsupseteq_F^{\mathsf{Eli}_\rho} E'$ *if and only if* $\exists b \in E'$ *s.t.* $\forall a \in E$ *we have* $a \succ_F^\rho b$

If there is an argument in E' that is ranked worse than any argument in E, then E is at least as plausible to be accepted as E'.

Proposition 90. Eli_ρ *is reflexive and transitive.* *Proof: P. 207*

Example 138. *Consider again AF F_4 from Example 6 and sets $\{a, b, g\}$,$\{a, c, g\}$, and $\{a, d, g\}$. If we consider the argument ranking induced by Cat, then argument c is ranked below a, b, d and g. Also, b is ranked below a, d and g, hence we have:*

$$\{a, d, g\} \sqsupset_{F_4}^{\mathsf{Eli}_{Cat}} \{a, b, g\} \sqsupset_{F_4}^{\mathsf{Eli}_{Cat}} \{a, c, g\}$$

The Elitist extension-ranking semantics can compare more sets than Kelly's extension-ranking semantics. As a reminder, for Kelly's extension-ranking semantics, the sets $\{a, c, g\}$ and $\{a, d, g\}$ are incomparable because they are not disjoint.

In Example 138, we see that the admissible set $\{a, c, g\}$ is not among the most plausible sets, and even worse, the conflict set $\{a, b, g\}$ is ranked strictly better. Consequently, *σ-generalisation* and *respect conflicts* are violated for Eli_{Cat} with $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Example 139. *Consider F_5 from Example 22 and the* h-categoriser argument-ranking semantics*. The set $\{h\}$ is the only most plausible non-empty set, every other set is ranked below it. The sets $\{j\}$ and $\{h, j\}$ are ranked above every set containing argument i. The extension ranking $\sqsupseteq_{F_5}^{\mathsf{Eli}_{Cat}}$ is depicted in Figure 8.30.*

In Example 139, we see that for any set only the lowest ranked argument according to Cat is relevant for extension-ranking semantics Eli_{Cat}. It is clear that this holds for any argument-ranking semantics ρ. Example 139 also shows that *weak reinstatement* is violated, since:

$$\{h\} \sqsupset_{F_5}^{\mathsf{Eli}_{Cat}} \{h, j\}$$

Hoare_ρ and Eli_ρ are syntactically similar, we can base many of our results for Eli_ρ on the results for Hoare_ρ. First, we show that *composition* is satisfied by Eli_ρ if ρ satisfies *In*.

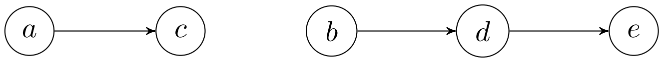

Figure 8.31: AF F_{43} from Example 140.

Proposition 91. *If ρ satisfies* In*, then* Eli_ρ *satisfies* composition. *Proof: P. 207*

For Eli_ρ we show that *decomposition* is violated if *VP* and *DP* are satisfied by ρ.

Example 140. *Let $F_{43} = (\{a,b,c,d,e\},\{(a,c),(b,d),(d,e)\})$ be an AF as depicted in Figure 8.31. Consider the sets $\{a,e\}$ and $\{b,c\}$. If argument-ranking semantics ρ satisfies* VP *and* DP*, then we have $a \succ^{\rho}_{F_{43}} c$ and $e \succ^{\rho}_{F_{43}} c$, hence:*

$$\{a,e\} \sqsupset^{\mathsf{Eli}_\rho}_{F_{43}} \{b,c\}$$

F_{43} can be partitioned into two disjoint AFs $F_{43,1} = (\{a,c\},\{(a,c)\}$ and $F_{43,2} = (\{b,d,e\},\{(b,d),(d,e)\})$. In $F_{43,2}$, we have $b \succ^{\rho}_{F_{43,2}} e$ and therefore:

$$\{b\} \sqsupset^{\mathsf{Eli}_\rho}_{F_{43,2}} \{e\}$$

Thus, decomposition *is violated by* Eli_ρ *if ρ satisfies* VP *and* DP.

To show that *addition robustness* is violated by Eli_{Cat}, we extend Example 106.

Example 141. *Let F_{44} be the AF depicted in Figure 8.32. The argument ranking induced by Cat is:*

$$a \simeq^{Cat}_{F_{44}} b \succ^{Cat}_{F_{44}} g \succ^{Cat}_{F_{44}} c \simeq^{Cat}_{F_{44}} d \simeq^{Cat}_{F_{44}} e \succ^{Cat}_{F_{44}} h \succ^{Cat}_{F_{44}} f$$

Consider sets $\{a,e\}$ and $\{b,h\}$. It holds that:

$$\{a,e\} \sqsupset^{\mathsf{Eli}_{Cat}}_{F_{44}} \{b,h\}$$

We add attack (a,b) to get F'_{44}, and the resulting argument ranking is:

$$a \succ^{Cat}_{F_{44}} g \succ^{Cat}_{F_{44}} c \simeq^{Cat}_{F_{44}} d \succ^{Cat}_{F_{44}} b \succ^{Cat}_{F_{44}} h \succ^{Cat}_{F_{44}} e \succ^{Cat}_{F_{44}} f$$

Therefore, we have $b \succ^{Cat}_{F_{44}} e$ and $h \succ^{Cat}_{F_{44}} e$, thus:

$$\{b,h\} \sqsupset^{\mathsf{Eli}_{Cat}}_{F'_{44}} \{a,e\}$$

This shows that addition robustness *is violated.*

Finally, *syntax independence* is satisfied by Eli_ρ if ρ satisfies *Abs*.

Proposition 92. *If ρ satisfies* Abs*, then* Eli_ρ *satisfies* syntax independence.
Proof: P. 207

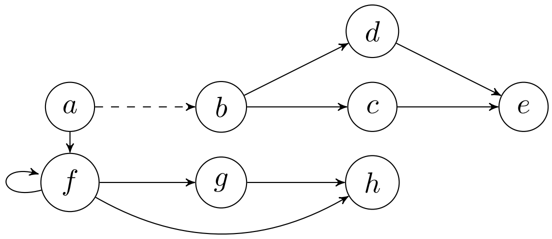

Figure 8.32: AF F_{44} from Example 141, where attack (a, b) is added later to obtain F'_{44}.

Principles	Eli_{Cat}
σ-generalisation	✗
respect conflicts	✗
composition	✓
decomposition	✗
weak reinstatement	✗
strong reinstatement	✗
addition robustness	✗
syntax independence	✓

Table 8.7: Principles and violated satisfied by Eli_{Cat}.

In summary, Eli_{Cat} satisfies *composition* and *syntax independence*, as shown in Table 8.7. Note that Eli_{Cat} and $\mathsf{GCB}_{Cat,\mathsf{min}}$ satisfy the same principle, which is not surprising since Eli_ρ is the strict variant of $\mathsf{GCB}_{\rho,\mathsf{min}}$ if ρ is a total argument-ranking semantics.

Proposition 93. *Let $F = (A, R)$ be an AF, $E, E' \subseteq A$, and ρ a total argument-ranking semantics, it holds that if and only if $E \sqsupset_F^{\mathsf{Eli}_\rho} E'$ then $E \sqsupset_F^{\mathsf{GCB}_{\rho,\mathsf{min}}} E'$.* Proof: P. 207

These two extension-ranking semantics agree on the strict parts of the extension rankings. The difference is that for two sets for which we cannot establish a strict relation, the Elitist extension-ranking semantics then says that these two sets are incomparable, whereas the general comparison-based extension-ranking semantics returns equally as plausible to be accepted. For example, in Example 139 we have $\{j\} \asymp_{F_5}^{\mathsf{Eli}_{Cat}} \{h, j\}$, while for $\mathsf{GCB}_{Cat,\mathsf{min}}$ we have $\{j\} \equiv_{F_5}^{\mathsf{GCB}_{Cat,\mathsf{min}}} \{h, j\}$.

At the beginning of this section, we discussed different ways of dealing with the empty set. We decided to state that the empty set is strictly more plausible than any non-empty set. In the proofs of the violation of the principles by the extension-ranking semantics Kelly_ρ, $\mathsf{Fishburn}_\rho$, and Eli_ρ, we saw that the choice of how to handle the empty set is not relevant. To show the violation, the empty set is not used. The only exception is *decomposition* for Kelly_ρ and $\mathsf{Fishburn}_\rho$. However, even in these cases, we have argued that the choice of how to handle the empty set is not crucial to the violation of *decomposition*. Therefore, we are open to placing the empty set somewhat freely in our rankings, depending on the situation.

This chapter shows that lifting operators and formal argumentation have been linked in the past, as discussed in Section 8.1, Section 8.2, and Section 8.3. In particular, the group comparison of Amgoud and Ben-Naim (2013) is a lifting operator used in formal argumentation. In Section 8.4, we discussed lifting operators for abstract argumentation in more detail. However, it turns out that simply applying lifting operators to an argument-ranking semantics is not sufficient to construct an extension-ranking semantics. We cannot construct an extension-ranking semantics that is a generalisation of Dung's extension semantics. In Example 128, we saw that there is no possible argument-ranking for F_{40} such that the most plausible sets are the admissible sets. Since all other lifting operators are generalisations of Kelly's extension, we are out of luck.

SOCIAL RANKING FUNCTIONS IN ABSTRACT ARGUMENTATION 9

In Chapter 8, we explored methods for translating *argument-ranking semantics* into *extension-ranking semantics*. In this chapter, we examine the reverse process: how to translate an extension-ranking semantics into an argument-ranking semantics, or more generally, defining a ranking over elements based on a ranking over sets of elements. This problem has been studied in detail in the field of *social ranking functions*. These functions were first introduced in the economics literature by Moretti and Öztürk (2017). These functions evaluate the performance of an individual based on the success of the groups they have been part of, providing a formal way to assess how well an individual can work in a team. In addition to economics, computer scientists have also studied social ranking functions (Bernardi et al., 2019; Khani et al., 2019; Haret et al., 2018a; Suzuki and Horita, 2024).

Definition 81. *Let S be a set and $\mathcal{R}$ a preorder on the powerset $\mathcal{P}(S)$. A* social ranking function *ξ maps $\mathcal{R}$ to a preorder on S.*[1]

In other words, a social ranking function maps a ranking over sets of arbitrary objects, such as players in a sports team, employees in a company, or arguments in an abstract argumentation framework, to a ranking over these objects. For example, if Peter is ranked higher than Karl, this means that Peter works better in different teams than Karl.

We begin this chapter by recalling a number of social ranking functions that can be found in the literature (Section 9.1). We then discuss some principles for social ranking functions that are important for our later investigation (Section 9.2). We end the chapter by introducing a new family of argument-ranking semantics by combining extension-ranking semantics and social ranking functions (Section 9.3).

[1]Note that we change the requirement of $\mathcal{R}$ from a total preorder to a preorder, and the output from a partial order to a preorder, to better suit our setting.

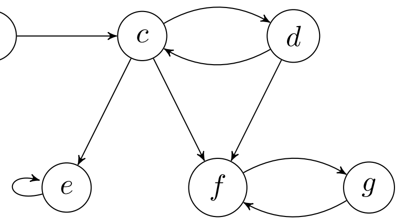

Figure 9.1: Recalled AF F_4 from Example 6.

9.1 SOCIAL RANKING FUNCTIONS

In this section, we recall several social ranking functions found in the literature.

9.1.1 Singleton Approach

The most direct way to define a social ranking function is to restrict the ranking over sets of objects to singleton sets. The behaviour of these singleton sets then gives us insight into the relationship between the objects. If $\{a\}$ is ranked better than $\{b\}$, then a is ranked better than b in the restricted ranking. Bernardi et al. (2019) proposed a social ranking function as follows:

Definition 82. *Let S be a set and $\mathcal{R}$ a preorder on the powerset $\mathcal{P}(S)$. For any two objects $x, y \in S$, the* singleton social ranking function $\mathsf{ST}_{\mathcal{R}}$ *is defined via:*

$$x \succeq^{\mathsf{ST}_{\mathcal{R}}} y \text{ if and only if } (\{x\}, \{y\}) \in \mathcal{R}$$

The singleton social ranking function is overly simplistic. Bernardi et al. (2019) discussed that a ranking based solely on singleton sets ignores all the information provided by rankings over sets with cardinality greater than one. This is also true in the context of abstract argumentation.

Example 142. *Consider AF F_4 from Example 6, as depicted again in Figure 9.1. Consider the* admissible extension-ranking semantics $\mathsf{r\text{-}ad}$*. The admissible sets are: $\{a\}$, $\{d\}$, and $\{g\}$, so:*

$$a \simeq_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} d \simeq_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} g$$

The sets $\{b\}$, $\{c\}$, and $\{f\}$ are not admissible but at least conflict-free and $\{e\}$ is not even conflict-free. Thus, the final ranking is:

$$a \simeq_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} d \simeq_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} g \succ_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} b \bowtie_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} c \bowtie_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} f \succ_{F_4}^{\mathsf{ST}_{\mathsf{r\text{-}ad}}} e$$

Example 142 shows the limitation of $\mathsf{ST}_{\mathsf{r\text{-}ad}}$. The induced preorder has at most three ranks. The first rank contains all arguments that defend themselves against any attacker, making the singleton set of these arguments admissible. The lowest rank contains every self-attacking argument. Between these ranks, we find all arguments that are not self-attacking but also not completely self-defending.

9.1.2 Lexicographic Excellence Operator

A more expressive social ranking function is the *lexicographic excellence operator* $\mathsf{lex\text{-}cel}_{\mathcal{R}}$, introduced by Bernardi et al. (2019). Elements are ranked according to the best sets in which they appear, and in the case of a tie, the ranking proceeds lexicographically. However, as proposed by Bernardi et al. (2019), the lexicographic excellence operator requires a total order of the sets as input, while our definition of a social ranking function requires only a preorder. To circumvent this problem, we define the notion of a *rank* of a set within a ranking to compare two sets.

Definition 83. *Let $X \subseteq S$ be a subset of S and $\mathcal{R}$ a preorder on the powerset $\mathcal{P}(S)$. Let $X_1, X_2, \ldots, X_k$ be a sequence such that $(X_1, X_2) \in \mathcal{R}, \ldots, (X_k, X) \in \mathcal{R}$. Then, we define the* rank of X, *as* $rank_{\mathcal{R}}(X) := k + 1$.

Additionally, for an element $x \in S$, we define:

$$x_{k,\mathcal{R}} := |\{X \in \mathcal{P}(S) \mid rank_{\mathcal{R}}(X) = k, x \in X\}|,$$

This represents the number of subsets on rank k that contain x.

With this definition, we define the *lexicographic excellence operator*.

Definition 84. *Let $x, y \in S$ be two elements of S and $\mathcal{R}$ a preorder over $\mathcal{P}(S)$. The* lexicographic excellence operator $\mathsf{lex\text{-}cel}_{\mathcal{R}}$ *is defined via:*

1. $x \succ^{\mathsf{lex\text{-}cel}_{\mathcal{R}}} y$ *if there exists a $k \in \mathbb{N}$ such that $x_{k,\mathcal{R}} > y_{k,\mathcal{R}}$ and for all $i < k$ $x_{i,\mathcal{R}} = y_{i,\mathcal{R}}$.*

2. $x \simeq^{\mathsf{lex\text{-}cel}_{\mathcal{R}}} y$ *if $x_{i,\mathcal{R}} = y_{i,\mathcal{R}}$ for all $i \in \mathbb{N}$.*

Intuitively, an object x is ranked better than y by the lexicographic excellence operator if x is contained in more higher-ranked sets than y.

Example 143. *Consider AF F_4 from Example 6 and the complete extension-ranking semantics* r-co*. F_4 has four complete extensions: $\{a\}$, $\{a, g\}$, $\{a, c, g\}$, and $\{a, d, g\}$. All of them are containing a, so $a_{1,\sqsupseteq^{\mathsf{r\text{-}co}}_{F_4}} = 4$. Argument b is part of none of these sets, so $b_{1,\sqsupseteq^{\mathsf{r\text{-}co}}_{F_4}} = 0$. Therefore, we get:*

$$a \succ^{\mathsf{lex\text{-}cel}_{\sqsupseteq^{\mathsf{r\text{-}co}}_{F_4}}} b$$

9.1.3 Ceteris Paribus Majority Solution

Another prominent social ranking function is the *Ceteris Paribus Majority Solution* $\mathsf{CPM}_{\mathcal{R}}$ by Haret et al. (2018a). This social ranking function compares objects based on their influence on the ranking of a set.

Definition 85. *Let S be a set and $\mathcal{R}$ a preorder on the powerset $\mathcal{P}(S)$. For any two objects $x, y \in S$, we define the* Ceteris Paribus Majority Solution $\mathsf{CPM}_{\mathcal{R}}$ *via:*

$$x \succeq^{\mathsf{CPM}_{\mathcal{R}}} y \text{ if and only if}$$

$$\Big|\{X \in \mathcal{P}(S \setminus \{x,y\}) \mid (X \cup \{x\}, X \cup \{y\}) \in \mathcal{R}\}\Big| \geq \Big|\{X \in \mathcal{P}(S \setminus \{x,y\}) \mid (X \cup \{y\}, X \cup \{x\}) \in \mathcal{R}\}\Big|$$

In other words, the Ceteris Paribus Majority Solution counts the number of times that a set of objects, containing neither x nor y, is ranked better after adding one of those objects.

Example 144. *Consider F_4 from Example 6 and the complete extension-ranking semantics* r-co. *To compare the two arguments a and b, we have to check, for each set that does not contain either of these arguments, whether these sets with a or b are more plausible to be accepted. For example, we must check if $\{a, c\}$ or $\{b, c\}$ is more plausible to be accepted. In this case, we get:*

$$\{a, c\} \sqsupset^{\text{r-co}}_{F_4} \{b, c\}$$

If we check this relationship for each set $X \subseteq A_4 \setminus \{a, b\}$, then we get 28 sets X for which $X \cup \{a\}$ is more plausible to be accepted than $X \cup \{b\}$ and no set X' for which $X' \cup \{b\}$ is more plausible to be accepted than $X' \cup \{a\}$. This implies:

$$a \succ^{\mathsf{CPM}_{\sqsupseteq^{\text{r-co}}_{F_4}}} b$$

9.1.4 Ordinal Banzhaf Solution

The last social ranking function we discuss is the *ordinal Banzhaf solution*, introduced by Khani et al. (2019). Similar to the Ceteris Paribus Majority Solution, the ordinal Banzhaf solution also considers the contribution of objects to a set in order to evaluate the individual objects. However, before we can define the corresponding social ranking function, we need some notations.

Definition 86. *Let S be a set and $\mathcal{P}(S)$ the corresponding powerset. For $x \in S$, we denote the set of subsets that do not contain x with $U_x(S) = \{X \subseteq \mathcal{P}(S) | x \notin X\}$.*

Definition 87. *Let S be a set and $\mathcal{R}$ a preorder on the powerset $\mathcal{P}(S)$. For an object $x \in S$ the* ordinal marginal contribution $m_x^{\mathcal{R}}(X)$ *of x to set $X \in U_x(S)$ with respect to $\mathcal{R}$ is defined as:*

$$m_x^{\mathcal{R}}(X) = \begin{cases} 1 & \text{if } (X \cup \{x\}, X) \in \mathcal{R}, \\ -1 & \text{if } (X, X \cup \{x\}) \in \mathcal{R}, \\ 0 & \text{otherwise.} \end{cases}$$

With $u_x^{+,\mathcal{R}}(S)$ we denote the set such that $m_x^{\mathcal{R}}(X) = 1$ and $u_x^{-,\mathcal{R}}(S)$ as the set where $m_x^{\mathcal{R}}(X) = -1$ for $X \subseteq U_x(S)$.

Example 145. *Consider F_4 from Example 6 and argument a. If we add argument a to set $\{c, g\}$, then we get a complete set. So with respect to the extension-ranking semantics* r-co*, we get:*

$$\{a, c, g\} \sqsupset^{\text{r-co}}_{F_4} \{c, g\}$$

Therefore, $m^{\text{r-co}}_a(\{c, g\}) = 1$.

However, argument a has a negative influence on the set $\{b, d\}$, since the addition of a to $\{b, d\}$ will result in a conflicting set. Hence, $m^{\text{r-co}}_a(\{b, d\}) = -1$.

Now we can define the ordinal Banzhaf solution.

Definition 88. *Let S be a set and $\mathcal{R}$ a preorder on the powerset $\mathcal{P}(S)$. For any two objects $x, y \in S$, we define the* ordinal Banzhaf solution $\text{Banz}_{\mathcal{R}}$ *via:*

$$x \succeq^{\text{Banz}_{\mathcal{R}}} y \textit{ if and only if } |u^{+,\mathcal{R}}_x(S)| - |u^{-,\mathcal{R}}_x(S)| \geq |u^{+,\mathcal{R}}_y(S)| - |u^{-,\mathcal{R}}_y(S)|$$

In other words, object x is ranked better than object y if the number of sets for which x has a positive influence minus the number of sets for which x has a negative influence is greater than the number of sets for which y has a positive influence minus the number of set y has a negative influence.

Example 146. *Continuing with Example 145. For argument a, we see that this argument has a positive influence on 32 sets and a negative influence on 32 sets, so:*

$$|u_a^{+,\sqsupseteq^{\text{r-co}}_{F_4}}(A_4)| - |u_a^{-,\sqsupseteq^{\text{r-co}}_{F_4}}(A_4)| = 0$$

Argument b has no positive influence on any set and negative influence on 60 sets, so:

$$|u_b^{+,\sqsupseteq^{\text{r-co}}_{F_4}}(A_4)| - |u_b^{-,\sqsupseteq^{\text{r-co}}_{F_4}}(A_4)| = -60$$

Thus, we get:

$$a \succ^{\text{Banz}_{\sqsupseteq^{\text{r-co}}_{F_4}}} b$$

9.2 PRINCIPLES OF SOCIAL RANKING FUNCTIONS

Similar to argument- and extension-ranking semantics, social ranking semantics have been studied axiomatically. In this section, we discuss some important principles.

The first principle we introduce is called *Independence from the worst set* and is part of the characterisation of the lexicographic excellence operator under the assumption that the ranking on the powerset is total (Bernardi et al., 2019). In our setting, we do not assume total preorders, so the characterisation results of Bernardi et al. (2019) do not hold in our setting. However it is easy to see that the lexicographic excellence operator still satisfies Independence from the worst set.

Definition 89 (Bernardi et al. (2019)). *Let S be a set, $\mathcal{R}$ a preorder over $\mathcal{P}(S)$, and $x, y \in S$. Let:*

$$w = \max_{X \in \mathcal{P}(S)} (\mathit{rank}_{\mathcal{R}}(X))$$

Assume that $\mathcal{R}'$ is another preorder on $\mathcal{P}(S)$ for which it holds:

- *$\mathit{rank}_{\mathcal{R}}(X) = \mathit{rank}_{\mathcal{R}'}(X)$ for all $X \in \mathcal{P}(S)$ such that $\mathit{rank}_{\mathcal{R}}(X) < w$.*
- *$\mathit{rank}_{\mathcal{R}'}(X) \geq w$ for all $X \in \mathcal{P}(S)$ such that $\mathit{rank}_{\mathcal{R}}(X) = w$*

Then, if the social ranking function ξ satisfies Independence from the worst set*, we must have that $x \succ^{\xi_{\mathcal{R}}} y$ implies $x \succ^{\xi_{\mathcal{R}'}} y$.*

In other words, if an object y is already ranked strictly worse than another x, and we further subdivide the worst sets, then y should still be strictly worse than x.

The next principle is inspired by the concept of *Pareto-efficiency* (Moulin, 2003).

Definition 90 (Bengel et al. (2024a, 2025)). *Let S be a set and $\mathcal{R}$ a preorder over $\mathcal{P}(S)$. Consider, $x, y \in S$ such that:*

- *$\mathit{rank}_{\mathcal{R}}(Z \cup \{x\}) \leq \mathit{rank}_{\mathcal{R}}(Z \cup \{y\})$ for all $Z \in \mathcal{P}(S)$ with $x, y \notin Z$;*
- *$\mathit{rank}_{\mathcal{R}}(Z \cup \{x\}) < \mathit{rank}_{\mathcal{R}}(Z \cup \{y\})$ for at least one $Z \in \mathcal{P}(S)$ with $x, y \notin Z$.*

Then, social ranking function ξ satisfies Pareto-efficiency *if $x \succ^{\xi_{\mathcal{R}}} y$.*

Intuitively, if every set with x is ranked better than the same set with y, then x should be ranked better than y.

The last principle we discuss is called *Dominating set* (Bengel et al., 2024a, 2025). This principle captures the intuition that if there is a set containing object x that is ranked better than every set containing object y, then x should be ranked better than y.

Definition 91 (Bengel et al. (2024a, 2025)). *Let S be a set and $\mathcal{R}$ a preorder on $\mathcal{P}(S)$. Consider $x, y \in S$ such that there exists $X \subseteq S$ with $x \in X$ and for all $Y \subseteq S$ with $y \in Y$, then $(X, Y) \in \mathcal{R}$ and $(Y, X) \notin \mathcal{R}$.*

A social ranking function ξ satisfies Dominating set *if and only if $x \succ^{\xi_{\mathcal{R}}} y$.*

Crucially, Independence from the Worst Set and Pareto-efficiency together imply Dominating set.

Proposition 94. *A social ranking function that satisfies* Independence from the Worst Set *and* Pareto-efficiency *also satisfies* Dominating set. *Proof: P. 208*

In this work, only the three principles discussed are relevant, but other principles can be found in the literature (Moretti and Öztürk, 2017; Khani et al., 2019; Haret et al., 2018a; Bernardi et al., 2019).

Next, we analyse the behaviour of the social ranking functions. Bernardi et al. (2019) have shown that the lexicographic excellence operator satisfies *Independence* *from the worst set*, while both Ceteris Paribus Majority Solution and ordinal the Banzhaf solution are violating *Independence from the worst set*, as shown by Haret et al. (2018a) and Khani et al. (2019), respectively. For the singleton social ranking function, we can show that *Independence from the worst set* is satisfied.

Proposition 95. *The singleton social ranking function satisfies Independence from the worst set.* *Proof: P. 208*

Since Ceteris Paribus Majority Solution and ordinal Banzhaf solution violate Independence from the worst set, and this principle is existential for our later discussion, we will not discuss the satisfied principles of these two social ranking functions further.

For the singleton social ranking function, *Pareto-efficiency* is violated.

Example 147. *Consider the set $\{a, x, y\}$ and a preorder $\sqsupseteq$ such that:*

$$\sqsupseteq: \emptyset = \{a\} = \{a, x\} \sqsupset \{x\} = \{y\} \sqsupset \{a, y\} = \{x, y\} = \{a, x, y\}$$

Then for every $Z \subseteq \{a, x, y\}$, we have rank$_{\sqsupseteq}(Z \cup \{x\}) \leq$ rank$_{\sqsupseteq}(Z \cup \{y\})$ and for $Z = \{a\}$, we have rank$_{\sqsupseteq}(\{a\} \cup \{x\}) <$ rank$_{\sqsupseteq}(\{a\} \cup \{y\})$. Therefore, we must have:

$$x \succ^{\mathsf{ST}_{\sqsupseteq}} y$$

However, since $\{x\} = \{y\}$, we have:

$$x \simeq^{\mathsf{ST}_{\sqsupseteq}} y$$

For the lexicographic excellence operator, we show that *Pareto-efficiency* is satisfied, and therefore because of Proposition 94, *Dominating set* is also satisfied.

Proposition 96. *Lexicographic excellence operator satisfies Pareto-efficiency.*
Proof: P. 209

9.3 GENERALISED SOCIAL RANKING ARGUMENT-RANKING SEMANTICS

A social ranking function takes a preorder over the powerset of objects and gives us a preorder on the objects themselves. Given that an extension ranking is a preorder over the powerset of arguments and an argument ranking is a preorder on arguments, a social ranking function can transform an extension ranking into an argument ranking. In Section 9.1, we discussed several social ranking functions. Instead of defining an argument-ranking semantics for each combination of extension-ranking semantics and social ranking function, we provide a general definition of a *social ranking argument-ranking semantics*.

Definition 92. *Let $F = (A, R)$ be an AF and ξ a social ranking function with respect to extension ranking τ. We call ξ_τ the* Social ranking argument-ranking semantics *such that for arguments $a, b \in A$:*

$$a \succeq_F^{\xi_\tau} b \text{ if and only if } a \succeq^{\xi_\tau} b$$

In other words, argument a is at least as strong as argument b if the social ranking function ξ applied to the extension ranking τ returns that a is ranked at least as good as b.

Example 148. *Consider AF F_4 from Example 6 and the complete extension-ranking semantics* r-co*.*

Set $\{a\}$ is a complete extension, while every other singleton set is not complete. The admissible sets $\{g\}$ and $\{d\}$ defend argument a, however $\{d\}$ also defends g, so:

$$\{g\} \sqsupset_{F_4}^{\mathsf{r\text{-}co}} \{d\}$$

The three conflict-free sets $\{b\}$, $\{c\}$, and $\{f\}$ are incomparable with respect to r-co*, while set $\{e\}$ is ranked worse than any other singleton set. Thus, the resulting argument ranking according to $\mathsf{ST}_{\mathsf{r\text{-}co}}$ is:*

$$a \simeq_{F_4}^{\mathsf{ST}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} g \succ_{F_4}^{\mathsf{ST}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} d \succ_{F_4}^{\mathsf{ST}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} b \bowtie_{F_4}^{\mathsf{ST}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} c \bowtie_{F_4}^{\mathsf{ST}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} f \succ_{F_4}^{\mathsf{ST}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} e$$

Next, we examine the lexicographic excellence operator. F_4 has four complete extensions $\{a\}$, $\{a,g\}$, $\{a,c,g\}$, and $\{a,d,g\}$. All of them contain a, so $a_{1,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 4$, three of these complete extension also contain g ($g_{1,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 3$), and for c and d we get $c_{1,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = d_{1,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 1$. Every other argument is not part of a complete extension, thus we are done with a and g. For the remaining arguments, we have to go to rank two. Here, we get $d_{2,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 2$, $c_{2,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 1$, $b_{2,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = f_{2,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = g_{2,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 0$. To compare b and f, we have to go to rank four, where we get $b_{4,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 2$ and $f_{4,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}} = 1$. So the complete ranking is:

$$a \succ_{F_4}^{\mathsf{lex\text{-}cel}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} g \succ_{F_4}^{\mathsf{lex\text{-}cel}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} d \succ_{F_4}^{\mathsf{lex\text{-}cel}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} c \succ_{F_4}^{\mathsf{lex\text{-}cel}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} b \succ_{F_4}^{\mathsf{lex\text{-}cel}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} f \succ_{F_4}^{\mathsf{lex\text{-}cel}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} e$$

For the Ceteris Paribus Majority Solution, we look at the arguments in pairs and calculate their relationship. Argument a is the best argument according to $\mathsf{CPM}_{\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}$. The corresponding scores can be found in the Appendix B in Table B.1. We present the induced argument ranking:

$$a \succ_{F_4}^{\mathsf{CPM}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} g \succ_{F_4}^{\mathsf{CPM}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} d \succ_{F_4}^{\mathsf{CPM}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} b \succ_{F_4}^{\mathsf{CPM}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} f \succ_{F_4}^{\mathsf{CPM}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} c \succ_{F_4}^{\mathsf{CPM}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} e$$

For the ordinal Banzhaf solution, we have to calculate for each argument x the values for $|u_x^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)|$ and $|u_x^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)|$:

$$|u_a^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| - |u_a^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| = 0 \qquad |u_b^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| - |u_b^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| = -60$$
$$|u_c^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| - |u_c^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| = -63 \qquad |u_d^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| - |u_d^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| = -38$$
$$|u_e^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| - |u_e^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| = -64 \qquad |u_f^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| - |u_f^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| = -64$$
$$|u_g^{+,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| - |u_g^{-,\sqsupseteq_{F_4}^{\mathsf{r\text{-}co}}}(A_4)| = -64$$

The resulting argument ranking is:

$$a \succ_{F_4}^{\mathsf{Banz}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} g \succ_{F_4}^{\mathsf{Banz}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} d \succ_{F_4}^{\mathsf{Banz}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} b \succ_{F_4}^{\mathsf{Banz}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} c \succ_{F_4}^{\mathsf{Banz}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} f \simeq_{F_4}^{\mathsf{Banz}_{\sqsupseteq^{\mathsf{r\text{-}co}}}} e$$

Next, we compare these argument rankings with the argument ranking with respect to the h-categoriser argument-ranking semantics with is:

$$a \succ_{F_4}^{Cat} g \succ_{F_4}^{Cat} d \succ_{F_4}^{Cat} e \succ_{F_4}^{Cat} b \succ_{F_4}^{Cat} c \succ_{F_4}^{Cat} f$$

All five argument rankings agree that argument a is a strong argument and therefore ranks high. In the argument rankings based on the social ranking function, the self-attacking argument e ranks poorly, while for Cat this argument ranks more towards the middle. We see that these five argument rankings are different.

Skiba et al. (2021b) discussed how to transform an extension ranking into an argument ranking. To be precise they present a variation of the lexicographic excellence operator to define an argument-ranking semantics. In that approach, an argument a is ranked better than argument b if there exists a set E containing a that is ranked better than any set containing b.

Definition 93 (Skiba et al. (2021b)). *Let $F = (A, R)$ be an AF, $a, b \in A$, and τ an extension-ranking semantics. We define an argument-ranking semantics $\succeq_F^\tau$ via:*

$$a \succeq_F^\tau b \text{ if and only if } \exists E \subseteq A \text{ with } a \in E \text{ such that } \forall E' \subseteq A \text{ with } b \in E'\text{: } E \sqsupseteq_F^\tau E'$$

Example 149. *Consider F_4 from Example 6 and the extension-ranking semantics* r-co. *F_4 has four complete extensions $\{a\}$, $\{a, g\}$, $\{a, c, g\}$, and $\{a, d, g\}$, implying that a, g, c, and d are all equally ranked and all are ranked better than b, f, and e. Since e is self-attacking, e is ranked below b and f. Also, b and f are incomparable to each other. Thus, the final argument ranking is:*

$$a \simeq_{F_4}^{\mathsf{r\text{-}co}} g \simeq_{F_4}^{\mathsf{r\text{-}co}} d \simeq_{F_4}^{\mathsf{r\text{-}co}} c \succ_{F_4}^{\mathsf{r\text{-}co}} b \bowtie_{F_4}^{\mathsf{r\text{-}co}} f \succ_{F_4}^{\mathsf{r\text{-}co}} e$$

Comparing the results of Examples 143, 148, and Example 149, we see that $\succeq_{F_4}^{\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}co}}}$ can distinguish between arguments a, g, d, and c, while for $\succeq_{F_4}^{\mathsf{r\text{-}co}}$ these four arguments are considered equally strong. Thus, $\succeq_{F_4}^{\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}co}}}$ can distinguish more arguments than $\succeq_{F_4}^{\mathsf{r\text{-}co}}$. In fact, the lexicographic excellence operator is more informative than the operator of Skiba et al. (2021b).

Proposition 97. *Let $F = (A, R)$ be an AF, $a, b \in A$, and τ an extension ranking. If $a \succeq_F^{\mathsf{lex\text{-}cel}_\tau} b$, then $a \succeq_F^\tau b$.* *Proof: P. 209*

We investigate the principles for social ranking argument-ranking semantics to understand which combinations of principles for extension-ranking semantics τ and social ranking function ξ are sufficient and necessary conditions for the corresponding social ranking argument-ranking semantics to satisfy the principles.

We begin by investigating the sufficient conditions for satisfying *σ-C*. To recall, *σ-C* states that credulously accepted argument with respect to an extension semantics σ should be ranked better than rejected arguments. We see that *Independence from the worst set* and *Pareto-efficiency* are sufficient to satisfy *σ-C* if the underlying extension-ranking semantics satisfies *σ-generalisation*.

Proposition 98. *Let $F = (A, R)$ be an AF, τ be an extension-ranking semantics satisfying* σ-generalisation *for extension semantics σ, and ξ be a social ranking function satisfying* Independence from the worst set *and* Pareto-efficiency. *Then the social ranking argument-ranking semantics ξ_τ satisfies* σ-C. *Proof: P. 210*

Next, we show that *Independence from the worst set* and *Pareto-efficiency* together imply that every skeptically accepted argument ranks above every non-skeptically accepted argument, which describes the principle *σ-sk-C*.

Proposition 99. *Let $F = (A, R)$ be an AF, τ be an extension-ranking semantics satisfying* σ-generalisation *for the extension semantics σ, and ξ be a social ranking function satisfying* Independence from the worst set *and* Pareto-efficiency. *Then the social ranking argument-ranking semantics ξ_τ satisfies* σ-sk-C. *Proof: P. 210*

Note that we cannot construct an abstract argumentation framework for every possible extension ranking. Any extension ranking for which we can find an argumentation framework is called *realisable*.

Definition 94. *Let E be a set of arguments and $\mathcal{R}$ be a preorder on $\mathcal{P}(E)$. We say that $\mathcal{R}$ is* τ-realisable *for an extension-ranking semantics τ if there is an AF $F = (A, R)$ with $A = E$ such that $\sqsupseteq^\tau_F = \mathcal{R}$.*

Example 150. *Consider* r-ad *and the preorder:*

$$\emptyset \equiv \{a, b, c\} \equiv \{a\} \equiv \{b\} \sqsupset \{a, b\} \sqsupset \{c\}$$

For $\{a\}$ and $\{b\}$ to be equally plausible to be accepted as $\emptyset$ according to r-ad*, it must hold that* $\mathsf{Conflicts}(\{a\}) = \mathsf{Conflicts}(\{b\}) = \emptyset$ *and* $\mathsf{Undef}(\{a\}) = \mathsf{Undef}(\{b\}) = \emptyset$. *Since $\{a, b, c\}$ is among the most plausible sets, a and b cannot attack each other, implying* $\mathsf{Conflicts}(\{a, b\}) = \emptyset$. *There must be an attacker of a or b which cannot be counterattacked, but we only have the three arguments a, b, and c, and since* $\mathsf{Conflicts}(\{a, b, c\}) = \emptyset$*, there cannot be an attack between these three arguments. Implying,* $\mathsf{Undef}(\{a, b\}) = \emptyset$*, and therefore there is no AF F such that:*

$$\emptyset \equiv^{\text{r-ad}}_F \{a, b, c\} \equiv^{\text{r-ad}}_F \{a\} \equiv^{\text{r-ad}}_F \{b\} \sqsupset^{\text{r-ad}}_F \{a, b\} \sqsupset^{\text{r-ad}}_F \{c\}$$

Thus, this extension ranking is not r-ad*-realisable.*

We can only find necessary conditions for the satisfaction of *σ-C* and *σ-sk-C* for extension rankings, which can be induced by an abstract argumentation framework. Next, we present the necessary conditions for ad-*C*.

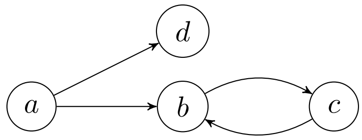

Figure 9.2: AF F_{45} from Example 151.

Proposition 100. *Let ξ be a social ranking such that $\xi_{\mathsf{r\text{-}ad}}$ satisfies* ad-C*. Then ξ satisfies* Dominating set *for all* r-ad*-realisable preorders $\sqsupseteq$.* *Proof: P. 210*

Propositions 98 and 100 suggest using a social ranking function that satisfies Independence from the worst set and Pareto-efficiency. The only social ranking function that does this is the lexicographic excellence operator. Thus, we only investigate the satisfied principles of $\mathsf{lex\text{-}cel}_\tau$ and leave the investigation of the remaining social ranking argument-ranking semantics to future work. Note that $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$ satisfies *σ-C* and *σ-sk-C* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

Next, we examine *Abstraction*, which is satisfied by $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Proposition 101. $\mathsf{lex\text{-}cel}_\tau$ *satisfies* Abs *for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.* *Proof: P. 211*

Independence is also satisfied by $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Proposition 102. $\mathsf{lex\text{-}cel}_\tau$ *satisfies* In *for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.* *Proof: P. 211*

Although *σ-C* and *σ-sk-C* are satisfied by $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$, *VP* is violated.

Example 151. *Consider the AF $F_{45} = (\{a, b, c, d\}, \{(a, b), (a, d), (b, c), (c, b)\})$ as depicted in Figure 9.2, and the corresponding (partial) extension ranking according to* r-ad *is:*

$$\{a, c\} \equiv^{\mathsf{r\text{-}ad}}_{F_{45}} \{a\} \equiv^{\mathsf{r\text{-}ad}}_{F_{45}} \{c\} \equiv^{\mathsf{r\text{-}ad}}_{F_{45}} \emptyset \sqsupset^{\mathsf{r\text{-}ad}}_{F_{45}} \{c, d\} \equiv^{\mathsf{r\text{-}ad}}_{F_{45}} \{d\} \asymp^{\mathsf{r\text{-}ad}}_{F_{45}} \{b\} \sqsupset^{\mathsf{r\text{-}ad}}_{F_{45}} \ldots$$

In this ranking, two sets of rank 1 contain arguments a and c. On rank 2, we have one set with c and no set with a, implying:

$$c \succ^{\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}ad}}}_{F45} a$$

However, argument a is unattacked, while argument c is attacked by b, thus VP *is violated. Since every remaining set containing a is not conflict-free, these sets are always ranked below $\{c, d\}$. Therefore, c is always ranked better than a with respect to* $\mathsf{lex\text{-}cel}_\tau$ *for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.*

At first glance it may seem counterintuitive that *VP* is violated. Both arguments a and c are skeptically accepted with respect to the complete extension semantics, so there is no reason to rank either argument worse. However, argument a is involved

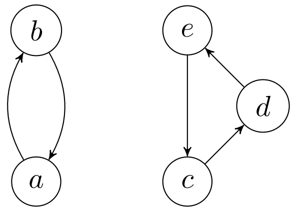

Figure 9.3: AF F_{46} from Example 152.

in more conflicts than c, and thus c is compatible with more arguments than a. We therefore argue that c should be ranked higher than a. In general, if we think back to the motivation of social ranking functions, employees who can work with more employees are considered better, so this ranking of a and c is in line with the idea behind social ranking functions.

For the principle *SC*, we can show a more general condition than just demonstrating that $\mathsf{lex\text{-}cel}_\tau$ satisfies it. We present a sufficient condition to satisfy *SC*.

Proposition 103. *Let $F = (A, R)$ be an AF. If the extension-ranking semantics τ satisfies* respect conflicts *and the social ranking function ξ satisfies* Dominating set*, then ξ_τ satisfies* SC. *Proof: P. 212*

The argument-ranking semantics $\mathsf{lex\text{-}cel}_\tau$ satisfies *SC* for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$. Consequently, *CP*, *CT*, and *SCT* are violated, as shown by Besnard et al. (2017).

Next, we show that *QP* is violated.

Example 152. *Let $F_{46} = (\{a, b, c, d, e\}, \{(a, b), (b, a), (c, d), (d, e), (e, c)\})$ as depicted in Figure 9.3. For $\mathsf{lex\text{-}cel}_\tau$ with $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$, the argument ranking is:*

$$a \simeq^{\mathsf{lex\text{-}cel}_\tau}_{F_{46}} b \succ^{\mathsf{lex\text{-}cel}_\tau}_{F_{46}} c \simeq^{\mathsf{lex\text{-}cel}_\tau}_{F_{46}} d \simeq^{\mathsf{lex\text{-}cel}_\tau}_{F_{46}} e$$

However, QP *would imply $c \succ_{F_{46}} a$ since $b \succ^{\mathsf{lex\text{-}cel}_\tau}_{F_{46}} e$, thus* QP *is violated by $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.*

Since *SC* is satisfied, *DP* cannot be satisfied.

Example 153. *Let $F_{47} = (\{a, b, c\}, \{(a, a), (c, b)\})$ be an AF as depicted in Figure 9.4, and ρ be any argument-ranking semantics. Assume ρ satisfies* SC *and* DP*. Since a is self-attacking, and if ρ satisfies* SC*, then it must hold that:*

$$b \succ^{\rho}_{F_{47}} a$$

However, $|a^-_{F_{47}}| = |b^-_{F_{47}}|$, $(a^-_{F_{47}})^- = a$, and $(b^-_{F_{47}})^- = \emptyset$, so if ρ satisfies DP*, it holds that:*

$$a \succ^{\rho}_{F_{47}} b$$

This is a contradiction, indicating that it is impossible to satisfy both SC *and* DP.

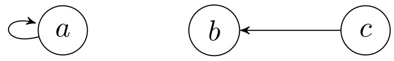

Figure 9.4: AF F_{47} from Example 153.

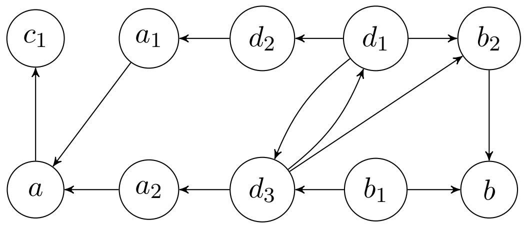

Figure 9.5: AF F_{48} from Example 154.

Note that the incompatibility between *SC* and *DP* has not been shown before, but since *CT* and *SCT* imply *DP* (Amgoud and Ben-Naim, 2013), this result is not surprising.

Next, we show that *DDP* is violated by lex-cel_τ for $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-gr}, \text{r-pr}, \text{r-sst}\}$.

Example 154. *Consider F_{48} as depicted in Figure 9.5. Then we have $|a^-_{F_{48}}| = |b^-_{F_{48}}| = 2$ and $|a^+_{F_{48}}| = |b^+_{F_{48}}| = 2$. The defence of a is simple and distributed, while the defence of b is simple and not distributed, since the attacker b_3 of b is attacked by two arguments. Consider the excerpt of the extension ranking of F_{48} with respect to* r-co *(for all ranks $i < 7$, we have no set S with $a \in S$ or $b \in S$):*

$$\begin{aligned} &\dots \\ 7: \quad &\{b\}, \{a_1, a_2, b, d_1, c_1\}, \{a_1, a_2, b_1\}, \{a_1, b_1, c_1\}, \\ &\{a_2, b_1, b_2\}, \{a_2, b_1, d_1\} \\ &\dots \end{aligned}$$

From this ranking, the induced ranking with respect to $\text{lex-cel}_{\text{r-co}}$ *gives us:*

$$b \succ^{\text{lex-cel}_{\text{r-co}}}_{F_{48}} a$$

Since $a_{i,\sqsupseteq^{\text{r-co}}_{F_{48}}} = b_{i,\sqsupseteq^{\text{r-co}}_{F_{48}}} = 0$ for all $i < 7$ and $a_{7,\sqsupseteq^{\text{r-co}}_{F_{48}}} = 0 < b_{7,\sqsupseteq^{\text{r-co}}_{F_{48}}} = 2$. Thus, b is ranked higher than a, even though its defence is not distributed. The same is true for all the other $\tau \in \{\text{r-ad}, \text{r-gr}, \text{r-pr}, \text{r-sst}\}$.

NaE is violated by lex-cel_τ for $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-gr}, \text{r-pr}, \text{r-sst}\}$.

Example 155. *Consider $F_{49} = (\{a, b, c\}, \{(b, c)\})$. Even though both a and b are unattacked, we get the ranking for $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-gr}, \text{r-pr}, \text{r-sst}\}$:*

$$a \succ^{\text{lex-cel}_\tau}_{F_{49}} b \succ^{\text{lex-cel}_\tau}_{F_{49}} c$$

The reason is that b is involved in more conflicts than a, so b is ranked lower.

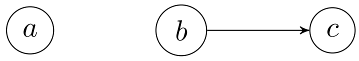

Figure 9.6: AF F_{49} from Example 155.

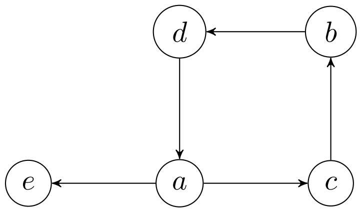

Figure 9.7: AF F_{50} from Example 156.

The example shows an interesting behaviour of $\mathsf{lex\text{-}cel}_\tau$: although both a and b are skeptically accepted with respect to complete extension semantics in F_{49}, it is still possible to distinguish these two arguments.

We show that *AvsFD* is satisfied by $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Proposition 104. $\mathsf{lex\text{-}cel}_\tau$ *satisfies* AvsFD *for* $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Proof: P. 212

The following example shows that *wσ-S* is violated by $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$ and $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

Example 156. *Consider AF* $F_{50} = (\{a,b,c,d,e\}, \{(a,c),(a,e),(c,b),(d,b),(d,a)\})$ *as depicted in Figure 9.7.*

In F_{50}*, argument* a *weakly* σ *supports* b *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$. *However, the extension ranking for* F_{50} *with respect to* $\mathsf{r\text{-}ad}$ *is depicted in Figure 9.8. The resulting ranking is:*

$$d \simeq^{\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}ad}}}_{F_{50}} c \succ^{\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}ad}}}_{F_{50}} e \succ^{\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}ad}}}_{F_{50}} b \succ^{\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}ad}}}_{F_{50}} a$$

The same holds for all extension-ranking semantics τ *with* $\tau \in \{\mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$. *Thus,* w$\sigma$-S *is violated for* σ *in* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

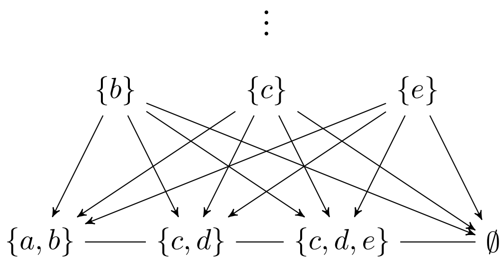

Figure 9.8: $\sqsupseteq^{\mathsf{r\text{-}ad}}_{F_{50}}$ from Example 156 depicted as a partial lattice.

	$\mathsf{lex\text{-}cel}_\tau$
Abs	✓
In	✓
VP	✗
SC	✓
CP	✗
QP	✗
CT	✗
SCT	✗
DP	✗
DDP	✗
NaE	✗
AvsFD	✓
σ-*C*	✓
σ-*sk*-*C*	✓
wσ-*S*	✗
sσ-*S*	✓
UR	✓(r-co)

Table 9.1: Principles satisfied and violated by $\mathsf{lex\text{-}cel}_\tau$ $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Unlike weak σ-Support, strong σ-support is satisfied by $\mathsf{lex\text{-}cel}_\tau$ for $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Proposition 105. $\mathsf{lex\text{-}cel}_\tau$ *satisfies* sσ-S *for* $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$ *and* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{sst}\}$. *Proof: P. 212*

We show that for any possible preorder for $\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}co}}$, we can find an abstract argumentation framework that induces that argument ranking.

Proposition 106. $\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}co}}$ *satisfies* UR. *Proof: P. 213*

In a recent paper, Leroy et al. (2024) have also discussed social ranking functions and abstract argumentation. They used a lexicographic combination of Dung's extension semantics to construct a preorder for the sets of arguments and then applied the lexicographic excellence operator to define an argument ranking. Their approach starts with stable sets, followed by preferred, complete, admissible, conflict-free, and ends with the remaining sets as equally ranked. Thus, they use the extension-ranking semantics:

$$\tau = \mathsf{lex}(\mathsf{LD}^{\mathsf{st}}, \mathsf{LD}^{\mathsf{pr}}, \mathsf{LD}^{\mathsf{co}}, \mathsf{LD}^{\mathsf{ad}}, \mathsf{LD}^{\mathsf{cf}})$$

They then apply $\mathsf{lex\text{-}cel}_\tau$ to this extension ranking. The problem with their extension ranking is that τ is not a generalisation of admissibility, and therefore ad-*C* is violated. In general, we cannot guarantee the satisfaction of σ-*C* for any $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

A related concept to social ranking functions is the *Shapley value*, introduced by Shapley (1953), which computes a fair distribution of gains or costs among a group of agents who have cooperated. An agent's Shapley value is the marginal contribution of each agent to all possible coalitions. Thus, the Shapley value is similar to *Ceteris Paribus Majority Solution* and *Banzhaf solution*. To the best of our knowledge, Shapley values have not been discussed as social ranking functions and therefore an axiomatic analysis is missing. Therefore, we leave the discussion of Shapley values as social ranking functions for future work.

In this chapter, we have discussed various social ranking functions and their use in formal argumentation. We proposed a new family of argument-ranking semantics called *social ranking argument-ranking semantics*, which applies a social ranking function to an extension ranking. It turns out that only the lexicographic excellence operator provides a well-behaved argument-ranking semantics. The results of our investigation are summarised in Table 9.1.

The resulting argument rankings provide a refinement of the acceptance of arguments, ranking all skeptically accepted arguments above all credulously accepted arguments, and these above all rejected arguments. Within these groups, we can still compare arguments, allowing us to say that one argument is stronger than another, even if both are skeptically accepted.

COMPUTATIONAL COMPLEXITY 10

So far, we have analysed extension-ranking semantics using a principle-based approach. These principles model various aspects of argumentative reasoning, and a well-behaved extension-ranking semantics should satisfy them. However, beyond these principles, it is also important to know how computational difficulty it is to compare two sets of arguments. This analysis is known as *computational complexity*, which determines how hard it is to solve a problem. In the next section (Section 10.1), we recall the necessary background information about computational complexity. In the following section (Section 10.2), we analyse our specific problems.

10.1 BACKGROUND

The main objective of *computational complexity theory* is to determine the *difficulty* of solving a problem. We first consider *decision problems* and their corresponding *complexity classes* (for an overview, see (Papadimitriou, 1994; Arora and Barak, 2009)). A decision problem takes a *instance* or *input* and returns `Yes` or `No` if that instance is a solution of a question. The complexity class **P** consists of decision problems that can be solved by a deterministic polynomial-time algorithm. These algorithms always producing the same output for a given input with a runtime that grows at most polynomially with the size of the input. The complexity class **NP** encompasses the decision problems where `Yes`-instances can be recognised by a non-deterministic polynomial-time algorithm. An algorithm is *non-deterministic* if it can have multiple different outcomes for the same input. For a complexity class $\mathcal{C}$, the *complement* class **co**$\mathcal{C}$ contains the problems for which the *complement problem* is in $\mathcal{C}$. For a problem $\mathcal{D}$, **co**$\mathcal{D}$ is the complement problem if every `Yes`-instance of $\mathcal{D}$ is a `No`-instance of **co**$\mathcal{D}$. For **NP**, **coNP** contains the problems where a `Yes`-instance of the complement can be recognised by an non-deterministic polynomial-time algorithm.

$\mathbf{NP}^{\mathcal{C}}$ denotes the complexity class consisting of decision problems that can be recognised by a non-deterministic polynomial-time algorithm with access to a $\mathcal{C}$-*oracle*. A $\mathcal{C}$-oracle is an abstract machine that can solve a problem in complexity class $\mathcal{C}$ in constant time. For **P** we can define an oracle access similarly, i. e. $\mathbf{P}^{\mathcal{C}}$ is the complexity class consisting of decision problems that can be solved by a deterministic polynomial-time algorithm with access to a *$\mathcal{C}$-oracle*. We define the *polynomial-time hierarchy* **PH** (Stockmeyer, 1976) as follows:

$$
\begin{aligned}
&\Sigma_0^p = \Pi_0^p = \Delta_0^p = \mathbf{P}\\
&\Sigma_1^p = \mathbf{NP}\\
&\Pi_1^p = \mathbf{coNP}\\
&\Sigma_{k+1}^p = \mathbf{NP}^{\Sigma_k^p}\\
&\Pi_{k+1}^p = \mathbf{coNP}^{\Sigma_k^p}\\
&\Delta_{k+1}^p = \mathbf{P}^{\Sigma_k^p}\\
&\bigcup_{k=0}^{\infty} \Sigma_k^p = \bigcup_{k=0}^{\infty} \Pi_k^p = \mathbf{PH}
\end{aligned}
$$

for $k > 0$. It is still an open whether **NP** = **coNP** holds (Papadimitriou, 1994; Arora and Barak, 2009). Note that if this is the case, then the entire polynomial-time hierarchy collapses to **NP**. While the relationship between **NP** and **coNP** is unclear, Papadimitriou and Yannakakis (1982) have introduced the complexity class **DP**, which consist out of decision problems that are the intersection of a decision problem in **NP** and a decision problem in **coNP**.

Another important notion is *deterministic polynomial-time reduction*. A decision problem $\mathcal{D}$ is *deterministic polynomial-time reducible* to a decision problem $\mathcal{D}'$ if there exists a deterministic polynomial-time algorithm $\mathcal{A}lg$ such that for every instance I of $\mathcal{D}$, I is a `Yes`-instance, if and only if $\mathcal{A}lg(I)$ is a `Yes`-instance of $\mathcal{D}'$. This allows us to transform any instance of $\mathcal{D}$ into an instance of $\mathcal{D}'$ with the same answer. We call a decision problem $\mathcal{D}$ *$\mathcal{C}$-hard* if every problem in $\mathcal{C}$ is deterministic polynomial-time reducible to $\mathcal{D}$. Furthermore, a decision problem $\mathcal{D}$ is *$\mathcal{C}$-complete* (short $\mathcal{C}$-c) if $\mathcal{D}$ is in $\mathcal{C}$ and $\mathcal{C}$-hard for a complexity class $\mathcal{C}$.

In abstract argumentation, we typically consider three decision problems: *verification*, *credulous acceptance* and *skeptical acceptance*. The verification problem $\textsc{Ver}_\sigma$ for extension semantics asks whether a set E is a σ-extension in abstract arugmentation framework F. Formally:

$\textsc{Ver}_\sigma$	**Input:**	An AF $F = (A, R)$ and $E \subseteq A$
	Output:	`Yes` if $E \in \sigma(F)$, and `No` otherwise

For several extension semantics, $\textsc{Ver}_\sigma$ is in **P** (Dung, 1995; Dimopoulos and Torres, 1996; Coste-Marquis et al., 2005; Dvorák and Woltran, 2011; Dvorák et al., 2012). However, $\textsc{Ver}_{\mathsf{pr}}$ is **coNP**-complete (Dunne and Bench-Capon, 2002), and $\textsc{Ver}_{\mathsf{sst}}$ is **coNP**-complete (Caminada et al., 2012; Dvorák and Woltran, 2010).

	cf	ad	co	gr	pr	st	sst
VER_σ	in **P**	in **P**	in **P**	in **P**	**coNP**-c	in **P**	**coNP**-c
CRED_σ	in **P**	**NP**-c	**NP**-c	in **P**	**NP**-c	**NP**-c	Σ_2^P-c
SKEP_σ	*trivial*	*trivial*	in **P**	in **P**	Π_2^P-c	**coNP**-c	Π_2^P-c

Table 10.1: Complexity results for Dung's extension semantics based on (Dvorák and Dunne, 2017).

For the two acceptance problems, we ask whether a given argument is credulously or skeptically accepted in an abstract argumentation framework with respect to an extension semantics σ.

CRED_σ **Input:** An AF $F = (A, R)$ and $a \in A$
Output: `Yes` if $a \in \mathsf{cred}_\sigma(F)$, and `No` otherwise

SKEP_σ **Input:** An AF $F = (A, R)$ and $a \in A$
Output: `Yes` if $a \in \mathsf{skep}_\sigma(F)$, and `No` otherwise

The acceptance problems are generally harder problems than the verification problem. For instance, CRED_σ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{st}\}$ is **NP**-complete. Note that for $\text{SKEP}_{\mathsf{ad}}$, the answer is trivially always `No`, since the empty set is always admissible.

The complexity results for the extension semantics $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ are recalled in Table 10.1. For a deeper discussion on computational complexity in abstract argumentation and more problems, we refer the reader to Dvorák and Dunne (2017).

Besides decision problems, there are also *counting problems* in complexity theory (Toda, 1991; Toda and Watanabe, 1992; Hemaspaandra and Vollmer, 1995; Durand et al., 2005). These counting problems ask "How many solutions can we find?". In this context, we discuss the *sharp-dot* ($\# \cdot \mathcal{C}$) complexity classes, which contain the problems that count the number of solutions a problem has, where the problem must be in complexity class $\mathcal{C}$. Formally:

Definition 95. *$\# \cdot \mathcal{C}$ is the class of functions f such that, for some predicate Q computable in $\mathcal{C}$ and some polynomial p, for every x, it holds that:*

$$f(x) = |\{y \mid p(|x|) = |y| \text{ and } Q(x, y)\}|$$

In other words, $\# \cdot \mathcal{C}$ is the class of functions that counts the number of objects that are solutions for a $\mathcal{C}$-predicate given an object x. Note that the sharp-dot complexity classes are different from the *sharp* complexity classes ($\#\mathcal{C}$) of Valiant (1979); only for $\mathcal{C} = \mathbf{P}$ do these two classes coincide ($\# \cdot \mathbf{P} = \#\mathbf{P}$ (Hemaspaandra and Vollmer, 1995)).

In abstract argumentation, we also have a few results on counting complexity problems. The first problem is the *extension counting problem* $\#\text{EXT}_\sigma$, which asks for the number of σ-extensions we can find within an abstract argumentation framework with respect to an extension semantics σ.

	ad	co	pr	st	sst
$\#\text{EXT}_\sigma$	$\# \cdot \mathbf{P}$	$\# \cdot \mathbf{P}$	in $\# \cdot \mathbf{coNP}$	$\# \cdot \mathbf{P}$	$\# \cdot \mathbf{coNP}$-c
$\#\text{CRED}_\sigma$	$\# \cdot \mathbf{P}$	$\# \cdot \mathbf{P}$	in $\# \cdot \mathbf{coNP}$	$\# \cdot \mathbf{P}$	$\# \cdot \mathbf{coNP}$-c

Table 10.2: Counting complexity results for Dung's extension semantics taken from (Fichte et al., 2019, 2024).

$\#\text{EXT}_\sigma$ **Input:** An AF $F = (A, R)$
Output: $|\sigma(F)|$

The second counting problem discussed is the *credulous counting problem*, which asks how many σ-extensions contain a given argument a.

$\#\text{CRED}_\sigma$ **Input:** An AF $F = (A, R)$ and $a \in A$
Output: $|\{E \in \sigma(F) \mid a \in E\}|$

The counting complexity results are summarised in Table 10.2 and are taken from (Fichte et al., 2019, 2024). Since the grounded extension semantics is unique, the corresponding counting problems are not interesting.

10.2 EXTENSION-RANKING PROBLEMS

In this section, we examine the extension ranking problem and analyse its computational complexity. The corresponding decision problem is defined as follows.

τ EXTENSION RANKING **Input:** An AF $F = (A, R)$ and $E, E' \subseteq A$
Output: `Yes` if $E \sqsupseteq^\tau_F E'$, and `No` otherwise

The extension ranking problem asks whether one set of arguments is at least as plausible to be accepted as another set. The complexity of this problem for an extension-ranking semantics provides insight into the difficulty of constructing the full extension ranking. If τ EXTENSION RANKING has a high complexity, then constructing the full extension ranking is also computationally hard.

To solve the extension ranking problem LD^σ EXTENSION RANKING for the least-discriminating extension-ranking semantics LD^σ, we determine whether the two non-identical sets are σ-extensions by comparing the outputs of VER_σ for E and E'. If E is a σ-extension, then E' cannot be strictly more plausible to be accepted than E with respect to LD^σ and if E' not a σ-extension, then the same holds. First, we investigate the extension semantics for which the verification problem is traceable.

Proposition 107. LD^σ EXTENSION RANKING *is in* **P** *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{st}\}$.
Proof: P. 216

For the two remaining extension semantics pr and sst, we show that our problem is **coDP**-complete. To show this statement we use the decision problem SAT-UNSAT.

SAT-UNSAT	**Input:**	Two propositional formulae φ, ψ.
	Output:	`Yes` if φ is satisfiable and ψ is unsatisfiable.

SAT-UNSAT is the canonical **DP**-complete problem (Papadimitriou and Yannakakis, 1982).

Proposition 108. LD^{σ} EXTENSION RANKING *is* **coDP**-*complete for* $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$.
Proof: P. 216

Since many of our extension-ranking semantics are based on the base relations proposed in Chapter 5, we first analyse these relations. We start with the set-based base relations: $\sqsupseteq^{\mathsf{Conflicts}}$, $\sqsupseteq^{\mathsf{Undef}}$, $\sqsupseteq^{\mathsf{Condef}}$ and $\sqsupseteq^{\mathsf{Unatt}}$. We can compare two sets of arguments efficiently with the base relations.

Proposition 109. $\sqsupseteq^{\tau}$ EXTENSION RANKING *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}, \mathsf{c\text{-}Conflicts}, \mathsf{c\text{-}Undef}, \mathsf{c\text{-}Condef}, \mathsf{c\text{-}Unatt}\}$ *is in* **P**.
Proof: P. 218

For the base relations $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}, \mathsf{c\text{-}Conflicts}, \mathsf{c\text{-}Undef}, \mathsf{c\text{-}Condef}, \mathsf{c\text{-}Unatt}\}$, we can decide with a deterministic polynomial-time algorithm whether one set of arguments is at least as plausible to accept as another. For $\mathsf{r}\text{-}\sigma$ and $\mathsf{r\text{-}c}\text{-}\sigma$ with $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$, we lexicographically combine the base relations. Thus, $\mathsf{r}\text{-}\sigma$ EXTENSION RANKING and $\mathsf{r\text{-}c}\text{-}\sigma$ EXTENSION RANKING with $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$ are in **P**.

Proposition 110. $\mathsf{r}\text{-}\sigma$ EXTENSION RANKING *and* $\mathsf{r\text{-}c}\text{-}\sigma$ EXTENSION RANKING *with* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$ *are in* **P**.
Proof: P. 219

Note that $\mathsf{r\text{-}pr}$ EXTENSION RANKING and $\mathsf{r\text{-}sst}$ EXTENSION RANKING being in **P** does not conflict with the fact that $\mathrm{VER}_{\mathsf{pr}}$ and $\mathrm{VER}_{\mathsf{sst}}$ are **coNP**-complete. To compare two sets E and E', we compare the base relations, which can be done efficiently, however to decide $E \in \sigma(F)$, we have to do more. For an extension-ranking semantics τ that satisfies *σ-generalisation*, the questions $E \in \mathtt{max}_{\tau}(F)$ and $E \in \sigma(F)$ for an abstract argumentation framework $F = (A, R)$ and $E \subseteq A$ are equivalent. So for a set E to be preferred it has to hold that $\mathsf{Conflicts}_F(E) = \mathsf{Undef}_F(E) = \emptyset$ and there is no strict superset E' satisfying the same. This maximisation gives us the increase in complexity. For $\mathsf{r\text{-}sst}$, we discussed in Example 62 that $\mathsf{Unatt}_F(E) \neq \emptyset$ does not imply that a set E is not a semi-stable extension. We have to show, that there is no set E', for which $\mathsf{Unatt}_F(E') \subset \mathsf{Unatt}_F(E)$. So, it is not easy to decide whether $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$ for an AF F without a stable extension.

For the two remaining base relations ($\tau \in \{\sqsupseteq^{\mathsf{nonatt}}, \sqsupseteq^{\mathsf{strdef}}\}$), we can also show that τ EXTENSION RANKING is in **P**.

Proposition 111. τ EXTENSION RANKING *for* $\tau \in \{\sqsupseteq^{\mathsf{nonatt}}, \sqsupseteq^{\mathsf{strdef}}\}$ *is in* **P**.
Proof: P. 219

10.2.1 Copeland-based Extension-ranking Semantics

To show the computational complexity of the family of Copeland-based extension-ranking semantics, we introduce the Copeland score in asymmetric form:

Definition 96. *Let $(\mathcal{A},\mathcal{N})$ be an election and $(\succsim_1,\ldots,\succsim_n)$ the votes of $\mathcal{N}$. The asymmetric Copeland Score $Copeland^+$ for alternative $x\in\mathcal{A}$ is defined as:*

$$Copeland^+(x)=\sum_{i=1}^{n}|\{y\in\mathcal{A}\mid x\succsim_i y, y\not\succsim_i x\}|+\frac{1}{2}|\{y\in\mathcal{A}\mid x\succsim_i y, y\succsim_i x\}|$$

The Copeland Rule $Copeland^+$ for two alternatives $x,y\in\mathcal{A}$ is:

$$x\geq^{Copeland^+} y \text{ if and only if } Copeland^+(x)\geq Copeland^+(y)$$

Let $F=(A,R)$ be an AF, $E,E'\subseteq A$, and $(\sqsupseteq_F^{\tau_1},\ldots,\sqsupseteq_F^{\tau_n})$ be a sequence of base relations. We define the asymmetric Copeland-based combination $\mathsf{cope}^+(\sqsupseteq_F^{\tau_1},\ldots,\sqsupseteq_F^{\tau_n})$ *via:*

$$E\sqsupseteq_F^{\mathsf{cope}^+(\tau_1,\ldots,\tau_n)} E' \text{ if and only if } E\geq^{Copeland^+} E'$$

In other words, an alternative scores one point for each strict win over another alternative and half a point for each draw. Although the scores differ, the resulting rankings are the same.

Proposition 112. *Let $F=(A,R)$ be an AF, $E,E'\subseteq A$, and $(\sqsupseteq_F^{\tau_1},\ldots,\sqsupseteq_F^{\tau_n})$ be a sequence of base relations, the following holds:*

$$E\sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E' \text{ if and only if } E\sqsupseteq_F^{\mathsf{cope}^+(\tau_1,\ldots,\tau_n)} E'$$

Proof: P. 220

We analyse the complexity of $Copeland^+(x)$ for a given vote, which involves counting the alternatives that x is strictly preferred to, and draws with.

$\#\text{COPELAND}_{\succsim}^{\text{WIN}}$	**Input:**	Alternatives $\mathcal{A}$ and $x\in\mathcal{A}$
	Output:	$\lvert\{y\in\mathcal{A}\mid x\succsim y, y\not\succsim x\}\rvert$

$\#\text{COPELAND}_{\succsim}^{\text{DRAW}}$	**Input:**	Alternatives $\mathcal{A}$ and $x\in\mathcal{A}$
	Output:	$\lvert\{y\in\mathcal{A}\mid x\succsim y, y\succsim x\}\rvert$

Note that $\#\text{COPELAND}_{\succsim}^{\text{WIN}}$ and $\#\text{COPELAND}_{\succsim}^{\text{DRAW}}$ are counting problems, not decision problems. Additionally, we need the corresponding decision problem to determine if $x\succsim y$ is decidable in complexity class $\mathcal{C}$.

$\succsim$ RANKING	**Input:**	Election $(\mathcal{A},\mathcal{N})$, vote $v_i\in\mathcal{N}$, and $x,y\in\mathcal{A}$
	Output:	`Yes` if $x\succsim_{v_i} y$, and `No` otherwise

Proposition 113. *If* $\succsim$ RANKING *is decidable in* $\mathcal{C}$*, then* $\#\text{COPELAND}^{\text{WIN}}_{\succsim}$ *and* $\#\text{COPELAND}^{\text{DRAW}}_{\succsim}$ *are in* $\# \cdot \mathcal{C}$. *Proof: P. 220*

We use the computational complexity of the underlying counting problems to determine the complexity of $\mathsf{cope}(\tau_1, \dots, \tau_n)$ EXTENSION RANKING

Proposition 114. *If* τ EXTENSION RANKING *for* $\tau \in \{\tau_1, \dots, \tau_n\}$ *is in* $\mathcal{C}$*, then* $\mathsf{cope}(\sqsupseteq^{\tau_1}, \dots, \sqsupseteq^{\tau_n})$ EXTENSION RANKING *is in* $\mathbf{P}^{\# \cdot \mathcal{C}}$. *Proof: P. 220*

If the corresponding decision problems for the base relations $\tau_1, \dots, \tau_n$ are decidable with a deterministic polynomial-time algorithm, then the computational complexity of $\mathsf{cope}(\sqsupseteq^{\tau_1}, \dots, \sqsupseteq^{\tau_n})$ EXTENSION RANKING is known. Thus, $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ EXTENSION RANKING, $\mathsf{cope}(\sqsupseteq^{\mathsf{nonatt}})$ EXTENSION RANKING, and $\mathsf{cope}(\sqsupseteq^{\mathsf{strdef}})$ EXTENSION RANKING are in $\mathbf{P}^{\# \cdot \mathbf{P}}$.

10.2.2 Conditional-based Extension-ranking Semantics

To decide if a set E is at least as plausible to be accepted as another set E' with respect to $\sqsupseteq^{\kappa^Z}$ in an abstract argumentation framework $F = (A, R)$, we compare the values $\kappa^Z_F(E)$ and $\kappa^Z_F(E')$. To compute these values, we identify the highest-ranked attack violated by these two sets. This requires computing the Z-attack-Partitioning of F. We decide if an attack $r \in R$ is tolerated by R. We define the corresponding decision problem as follows:

TOLERATION	**Input:**	$F = (A, R)$ and $r \in R$
	Output:	`Yes` if r is tolerated by R, and `No` otherwise

For a longer discussion on computational complexity for conditional logic, see the work by Eiter and Lukasiewicz (2000).

Lemma 7. TOLERATION *is* **NP**-*complete.* *Proof: P. 221*

To decide if one attack is tolerated by all attacks in an abstract argumentation framework, we start with R_0. To compute R_0, we solve the problem TOLERATION for every attack in R. If the output is `Yes`, we add the attack to R_0. For each subsequent layer R_i, we solve the problem TOLERATION for all remaining attacks $R \setminus (R_0 \cup \dots \cup R_{i-1})$ in the restricted abstract argumentation framework $F = (A, R \setminus \{R_0 \cup \dots \cup R_{i-1}\})$. If the output is `Yes`, we add the attack to R_i.

Lemma 8. *Let* $F = (A, R)$ *be an AF and* $(R_0, \dots, R_n)$ *the Z-attack-Partitioning of* F. *For* $0 \leq i \leq n$ $R_i = R'_0$ *for Z-attack-Partitioning* $(R'_0, \dots, R'_m)$ *of* $F' = (A, R \setminus \{R_0 \cup \dots \cup R_{i-1}\})$. *Proof: P. 221*

Now that we understand the complexity of computing the Z-attack-Partitioning of F, we determine the computational complexity of $\sqsupseteq^{\kappa^Z}$ EXTENSION RANKING.

Proposition 115. $\sqsupseteq^{\kappa^Z}$ EXTENSION RANKING *is in* Δ^p_2. *Proof: P. 221*

10.2.3 General Comparison-based Extension-ranking Semantics

To determine the complexity of $\mathsf{GCB}_{\rho,\Box}$ EXTENSION RANKING, we need to examine the argument-ranking semantics and the comparison relation. We will first look at these two elements independently and then investigate the combination.

We start with the comparison relation $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. These relations take two vectors as input and compare the values to determine whether one set of arguments is at least as plausible to be accepted as another. All the comparison relation we have discussed are computable by a deterministic polynomial-time algorithm.

Proposition 116. *Comparison relations $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$ are computable by a deterministic polynomial-time algorithm.* *Proof: P. 222*

Next, we consider the argument-ranking semantics ρ. We introduce the corresponding decision problem for argument rankings, presented as generally as possible:

ρ ARGUMENT RANKING	**Input:**	An AF $F = (A, R)$ and $a, b \in A$
	Output:	`Yes` if $a \succeq^{\rho}_{F} b$, and `No` otherwise

In other words, we want to determine the computational complexity of comparing the strength of two arguments. To the best of our knowledge, the only ranking-based semantics for which the computational complexity is known is the serialisability argument-ranking semantics by Blümel and Thimm (2022), for all other argument-ranking semantics the computational complexity is unknown, including gradual semantics.

Since we can compute the comparison relations with a deterministic polynomial-time algorithm but lack results for argument-rankings semantics, we can only provide a parameterised representation of the computational complexity of the decision problem $\mathsf{GCB}_{\rho,\Box}$ EXTENSION RANKING.

Proposition 117. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* $\mathsf{GCB}_{\rho,\Box}$ EXTENSION RANKING *is decidable in $\mathbf{P}^{\mathcal{C}}$ for $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.* *Proof: P. 222*

For gradual semantics ne_σ with $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$, we need to solve the counting problem $\#\mathrm{CRED}_\sigma$ for every argument in the abstract argumentation framework. Thus, $\mathsf{GCB}_{ne_\sigma,\Box}$ EXTENSION RANKING for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{st}\}$ is in $\mathbf{P}^{\#\cdot\mathbf{P}}$ and for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$ in $\mathbf{P}^{\#\cdot\mathbf{coNP}}$, as shown by Fichte et al. (2024).

Proposition 118. $\mathsf{GCB}_{ne_\sigma,\Box}$ EXTENSION RANKING *for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{st}\}$ is in $\mathbf{P}^{\#\cdot\mathbf{P}}$ and for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$ in $\mathbf{P}^{\#\cdot\mathbf{coNP}}$ for $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.* *Proof: P. 222*

10.2.4 Lifting Operators

The final family of extension-ranking semantics is based on lifting operators. We discuss the computational complexity of Hoare_ρ, Kelly_ρ, $\mathsf{Fishburn}_\rho$, and Eli_ρ. Since these four extension-ranking semantics rely on an underlying argument-ranking semantics ρ, we can only give a parameterised complexity for the corresponding problems. We examine the lifting operators and show that they can be computed by a deterministic polynomial-time algorithm. We model the underlying argument-ranking semantics ρ as an oracle function that decides whether one argument is stronger than another. Given two sets of arguments E and E', we analyse the computational complexity of each lifting operator.

To decide $E \sqsupseteq_F^{\mathsf{Hoare}_\rho} E'$ in an abstract argumentation framework $F = (A, R)$ with respect to argument-ranking semantics ρ, we need to find a stronger argument in E for each argument in E'.

Proposition 119. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* Hoare_ρ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$. *Proof: P. 223*

To decide $E \sqsupseteq_F^{\mathsf{Kelly}_\rho} E'$ in an abstract argumentation framework $F = (A, R)$ with respect to argument-ranking semantics ρ, we need to check whether every argument of E is stronger than every argument of E'.

Proposition 120. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* Kelly_ρ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$. *Proof: P. 223*

Fishburn's extension-ranking semantics needs more checks than Kelly's extension-ranking semantics. Any argument only in E must be stronger than any argument only in E', and arguments in both sets must be stronger than any argument in E' and weaker than any argument in E.

Proposition 121. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* $\mathsf{Fishburn}_\rho$ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$. *Proof: P. 223*

For the Elitist extension-ranking semantics, we need to find an argument in E' that is ranked worse than every argument in E.

Proposition 122. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* Eli_ρ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$. *Proof: P. 224*

Extension ranking semantics τ	τ Extension Ranking
LD^σ for $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{st}\}$	in $\mathbf{P}$
LD^σ for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$	**coDP**-complete
$\sqsupseteq^{\text{(c-)Conflicts}}$	in $\mathbf{P}$
$\sqsupseteq^{\text{(c-)Undef}}$	in $\mathbf{P}$
$\sqsupseteq^{\text{(c-)Condef}}$	in $\mathbf{P}$
$\sqsupseteq^{\text{(c-)Unatt}}$	in $\mathbf{P}$
$\sqsupseteq^{\text{nonatt}}$	in $\mathbf{P}$
$\sqsupseteq^{\text{strdef}}$	in $\mathbf{P}$
r-σ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$	in $\mathbf{P}$
r-c-σ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$	in $\mathbf{P}$
$\mathsf{cope}(\sqsupseteq^{\text{Conflicts}}, \sqsupseteq^{\text{Undef}})$	in $\mathbf{P}^{\#\cdot\mathbf{P}}$
$\mathsf{cope}(\sqsupseteq^{\text{nonatt}})$	in $\mathbf{P}^{\#\cdot\mathbf{P}}$
$\mathsf{cope}(\sqsupseteq^{\text{strdef}})$	in $\mathbf{P}^{\#\cdot\mathbf{P}}$
$\sqsupseteq^{\kappa^Z}$	in Δ_2^p
$\mathsf{GCB}_{\rho,\Box}$	in $\mathbf{P}^{\mathcal{C}}$
$\mathsf{GCB}_{ne_\sigma,\Box}$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{st}\}$	in $\mathbf{P}^{\#\cdot\mathbf{P}}$
$\mathsf{GCB}_{ne_\sigma,\Box}$ for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$	in $\mathbf{P}^{\#\cdot\mathbf{coNP}}$
Hoare_ρ	in $\mathbf{P}^{\mathcal{C}}$
Kelly_ρ	in $\mathbf{P}^{\mathcal{C}}$
$\mathsf{Fishburn}_\rho$	in $\mathbf{P}^{\mathcal{C}}$
Eli_ρ	in $\mathbf{P}^{\mathcal{C}}$

Table 10.3: Complexity results for extension-ranking semantics τ.

In this chapter, we analysed the computational complexity of comparing two sets of arguments. The results are summarised in Table 10.3. We found that for several extension-ranking semantics, such as the family of extension-ranking semantics using lexicographic combination, we can compare two sets of arguments efficiently. However, other extension-ranking semantics are harder to compute. For many of theses semantics, we present only the membership results, leaving the hardness results to future work. This is primarily because we lack computational complexity results for the underlying argument-ranking semantics. For extension-ranking semantics based on an argument-ranking semantics, we have presented parameterised results. If the computational complexity of the argument-ranking semantics is known, we can determine the computational complexity of the corresponding extension-ranking semantics. Notably, for the extension-ranking semantics $\mathsf{cope}(\sqsupseteq^{\mathsf{Aqc}_{Cat}})$, we cannot provide a computational complexity result, since the complexity of computing the h-categoriser argument-ranking semantics is unknown.

CONCLUSION 11

This chapter recaps the thesis's contributions, explores previously undiscussed related work, and proposes ideas for future research.

11.1 SUMMARY

The primary goal of this thesis was to rank sets of arguments based on their acceptability in abstract argumentation. We explored various methods for ranking sets of arguments and introduced the notion of *extension-ranking semantics*, which allows us to compare the plausibility of acceptance of different sets. We also examined how extension-ranking semantics relates to other reasoning approaches in abstract argumentation, as shown in Figure 11.1.

Chapter 3 introduces the general notion of extension-ranking semantics and its connection to Dung's extension semantics (Dung, 1995). Dung's semantics can be viewed as a specific instance of extension-ranking semantics, bridging the gap between extension-based and extension-ranking semantics.

Chapter 4 outlines several principles for extension-ranking semantics, each representing desirable behaviours. The first principle, σ-*generalisation*, requires that

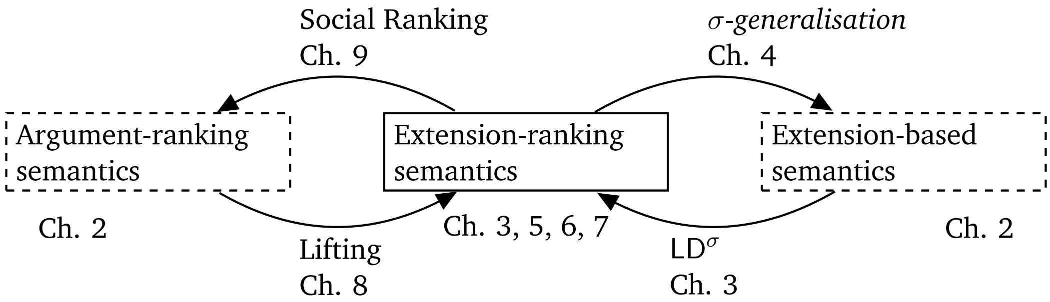

Figure 11.1: Thesis overview. Showing the connections between argument-ranking, extension-based, and extension-ranking semantics, where the dashed parts are not original content of this thesis. Each edge is discussed in a separate chapter.

extension-ranking semantics should generalise Dung's extension semantics. Any semantics satisfying this principle addresses our research question Q1 and connects extension-ranking semantics to extension-based semantics. Other principles examined include *respect conflicts*, *composition*, *decomposition*, *weak reinstatement*, *strong reinstatement*, *addition robustness*, and *syntax independence*, all relevant to research question Q3.

Chapter 5 defines several *base relations*, each capturing a specific aspect of argumentative reasoning. These include Conflicts, Undef, Condef, Unatt, nonatt, and strdef. While these base relations provide valuable insights, they do not fully capture argumentative reasoning on their own.

Chapter 6 builds on the previous chapter by combining base relations to define various extension-ranking semantics. We explore two combination methods: *lexicographic combination* and *Copeland-based combination*. By combining base relations, we generalise Dung's extension semantics. For instance, combining base relations Conflicts and Undef using lexicographic combination generalises admissibility. We also present generalisations of complete, grounded, preferred, and semi-stable extension semantics. These extension-ranking semantics allow us to determine not just acceptance or non-acceptance, but also the degree to which a set is "closer" to being admissible, complete, etc., addressing research questions Q1 and Q2. We evaluate these semantics based on the principles they satisfy, relating to research question Q3.

Chapter 7 introduces another family of extension-ranking semantics, drawing on concepts from *conditional logic* (Nute, 1980, 1984). We interpret an attack (a, b) between two arguments a an b as a conditional of the form "if argument a is accepted, then argument b is rejected". We define *ordinal conditional functions* (Spohn, 1988) within the context of abstract argumentation and introduce the notion of a *Z-attack-Partitioning* of an abstract argumentation framework. Inspired by *System Z* (Goldszmidt and Pearl, 1996), we develop an extension-ranking semantics and analyse its behaviour. Although this semantics is not a generalisation of Dung's extension semantics, it compares non-extensions well and identifies the best-ranked set satisfying given (nonsensical) constraints, thereby addressing research question Q2.

Chapter 8 explores the relationship between argument-ranking semantics and extension-ranking semantics. We discuss various methods to transform an argument ranking into an extension ranking, some inspired by definitions in formal argumentation literature (Aqc (Bonzon et al., 2018), Hoare_ρ (Amgoud and Ben-Naim, 2013) and $\mathsf{GCB}_{\epsilon,\Box}$ (Konieczny et al., 2015)) and others adapted from *lifting operators* (Barberà et al., 2004). While it is possible to lift argument rankings to extension rankings, the resulting ranking has some shortcomings. We cannot ensure that conflict-free sets are ranked better than conflicting ones.

Chapter 9 investigates the reverse process to Chapter 8: transforming an extension ranking into an argument ranking. This problem, known as *social rankings* in computational social choice (Moretti and Öztürk, 2017), involves ranking individual objects based on the ranking of sets of objects. We apply social ranking approaches from the literature to abstract argumentation and present sufficient and necessary conditions for these functions to induce well-behaved argument-ranking semantics. Together, Chapter 8 and Chapter 9 address research question Q4.

Chapter 10 focuses on research question Q5, examining the computational complexity of the extension-ranking problem. Specifically, we determine the difficulty of deciding whether one set is at least as plausible to be accepted as another set under various extension-ranking semantics. For some semantics, this problem is decidable by deterministic polynomial-time algorithm, while for others, we provide parametrised complexity classes.

11.2 RELATED WORK

This section discusses related works that were not covered in previous chapters.

A concept related is *approximation*, which measures how *close* a solution is to an optimal one. Several studies have explored approximation in abstract argumentation (Craandijk and Bex, 2020; Delobelle et al., 2023; Kuhlmann and Thimm, 2019; Malmqvist et al., 2020; Malmqvist, 2022; Thimm, 2021). Recent editions of the *International Competition of Computational Models of Argumentation* (ICCMA)[1] have also featured *approximation tracks*. These works aim to develop *heuristic algorithms* to solve decision problems like $\textsc{Cred}_\sigma$, which involves determining if a given argument is part of at least one σ-extension. This decision problem is **NP**-complete, making heuristic approaches valuable for speeding up computations. The approximation tracks involve solving instances of abstract argumentation frameworks and arguments, with algorithms assigning probabilities to argument sets being σ-extension, effectively creating a ranking over sets of arguments.

In theoretical computer science, approximation algorithms refer to those algorithms that solve *optimisation problems* with a guaranteed *approximation quality* (Vazirani, 2001). Works by Thimm (2024) and Skiba and Thimm (2024) discuss this interpretation of approximation in abstract argumentation, presenting soft notions of stable semantics and admissibility to define optimisation problems and show the theoretically guaranteed approximation quality. These two papers provide measures of how close a set is to being stable or admissible, allowing sets to be ranked based on this closeness.

This thesis utilises ideas and approaches from *computational social choice*, such as Copeland-based combination, lifting operators or social ranking functions. Several works discuss the connection between argumentation and computational social choice (Rahwan and Tohmé, 2010; Caminada and Pigozzi, 2011; Delobelle et al., 2018, 2016; Baumeister et al., 2021; Bernreiter et al., 2024; Awad et al., 2017; Chen and Endriss, 2017). The two papers (Rahwan and Tohmé, 2010) and (Caminada and Pigozzi, 2011) explore *judgment aggregation* (List and Pettit, 2002; List et al., 2009) in abstract argumentation. This area of research is concerned with finding a collective position for agents with different opinions. For example, a group of judges should

[1] http://argumentationcompetition.org/

give a collective verdict. To find such a collective position, the opinions are aggregated, but Arrow (1963) has already shown that such aggregation is not always possible. Rahwan and Tohmé (2010) and Caminada and Pigozzi (2011) looked at how to aggregate different labellings for arguments in an abstract argumentation framework into a single labelling. These two papers present and analyse approaches to aggregate labellings. Since labellings and extensions are related (Caminada, 2006), it is of interest to investigate the aggregation of extensions and extension-rankings as well.

Bernreiter et al. (2024) discussed a problem related to aggregation of labellings. They extend an abstract argumentation framework by incorporating a finite set of voters and ballots about the acceptance of arguments, to find the best set of arguments to represent the views of the voters. For each set of arguments, the authors compute a score for how well that set represents the voters' view. The authors show that it is not always possible to find a perfect fit for all voters, therefore they propose methods to find an optimal set in this situation. The difference between the approaches of Bernreiter et al. (2024) and extension-ranking semantics is that Bernreiter et al. (2024) needs the ballots as input, while for an extension-ranking semantics an abstract argumentation framework is sufficient.

Already Aristotle categorises arguments into *logic*, *dialectic*, and *rhetoric* types (van Eemeren et al., 2014). Logical strength of an argument can be divided into two parts *inferential* and *contextual* strength. The inferential strength is concerned with how well the premise of an argument supports its conclusion. Contextual strength is concerned with how well the conclusion is supported in the context. With dialectic strength we identify how attackable an argument is in the context of a discussion. The final notion of rhetorical strength is concerned with how well an argument can persuade an agent. Prakken's recent works (Prakken, 2021, 2022, 2023, 2024) discuss these strength notions, arguing that formal argumentation typically focuses on *contextual* strength. The strength of an argument is evaluated in the context of the abstract argumentation framework. Prakken discusses what dialectical strength looks like in formal argumentation. The author presents an argument-ranking semantics and investigates the properties of this semantics. It turns out that the principles from Definition 22 are only concerned with the contextual strength of arguments. In this thesis, we interpret the strength of arguments and sets of arguments as *contextual*, since this is the most appropriate interpretation in our context. The strength of arguments and sets of arguments should be evaluated in the context of an abstract argumentation framework.

11.3 FUTURE WORK

Throughout this thesis, several ideas for future research have been discussed. One open question is determining the complexity results for the argument-ranking semantics ρ. As discussed in Chapter 10, understanding the computational complexity of the argument-ranking semantics is essential for fully investigating the complexity of their corresponding extension-ranking semantics.

Another open problem is the *realisability* question for extension-ranking semantics: given an extension ranking, does there exists an abstract argumentation framework that induces it? For Dung's extension semantics, it is known under which conditions we can find an AF that induces a set of σ-extensions (Dunne et al., 2015, 2014; Linsbichler et al., 2016b,a). For several argument-ranking semantics, this question is trivial Yes, and a corresponding AF can be constructed (Skiba et al., 2022; Oren and Yun, 2023; Mailly, 2023). Based on these realisability results, a new notion of equivalence for AFs can be defined, where two AFs are equivalent if they induce the same extension ranking with respect to an extension-ranking semantics. This notion can provide deeper insights into the behaviour of these semantics and allow comparisons with other proposed equivalence notions (Oikarinen and Woltran, 2010, 2011; Baumann, 2012; Baumann et al., 2022, 2023; Bengel et al., 2024b). For example, both r-pr and r-co-pr are generalisations of the preferred extension semantics, however, these two extension-ranking semantics induce different rankings over the sets of arguments. So, while deciding the usage of either of these semantics, we have to consider which features are more important. So, a deeper investigation of the nuance behaviours of these two semantics would be interesting.

An application of extension-ranking semantics is *extension enforcement* (Baumann et al., 2021; Wallner, 2020, 2018; Wallner et al., 2016, 2017; Coste-Marquis et al., 2015; Haret et al., 2018b). The goal is to *modify* the abstract argumentation framework so that a selected set becomes a σ-extension in the modified AF. Thimm (2024) suggested that a set of arguments "close" to being stable might be easier to enforce than a less "close" set. Extension-ranking semantics can model this notion of *closeness*, with better-ranked sets being "closer" to acceptance. Investigating whether the assumption of Thimm (2024) holds is an interesting avenue for future research.

Our proposed base relations can help identify reasons for the non-acceptance of sets of arguments. For instance, if the base relation Undef highlights an argument a for a set E, it indicates that E does not defend a, *explaining* why E cannot be admissible. This aligns with the growing interest in *explanations* for AI approaches, known as *XAI* (Samek et al., 2017; Adadi and Berrada, 2018; Guidotti et al., 2019). Explanations of formal argumentation have also gained attention (Timmer et al., 2015; Potyka et al., 2023; Seselja and Straßer, 2013; Fan and Toni, 2014, 2015; Booth et al., 2014; Sakama, 2018; Baumann and Ulbricht, 2018; Saribatur et al., 2020; Borg and Bex, 2021a,b, 2022; Alfano et al., 2020, 2023; Cyras et al., 2021). It would be valuable to compare explanations based on our base relations with other explanation approaches and explore how extension rankings can provide explanations for settings with nonsensical constraints, offering more informative answers than mere rejection.

APPENDIX A

A.1 TECHNICAL PROOFS OF CHAPTER 3

Proposition 1. LD^σ *is reflexive and transitive for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof. Let $F = (A, R)$ be an AF, $E \subseteq A$, and $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ an extension semantics.

reflexive By definition, $E \equiv_F^{\mathsf{LD}^\sigma} E'$ for $E = E'$. This holds because if $E \in \sigma(F)$, then $E' \in \sigma(F)$ and if $E \notin \sigma(F)$, then $E' \notin \sigma(F)$.

transitive Consider $E', E'' \subseteq A$ such that $E \sqsupseteq_F^{\mathsf{LD}^\sigma} E'$ and $E' \sqsupseteq_F^{\mathsf{LD}^\sigma} E''$. If $E \notin \sigma(F)$, we know that $E' \notin \sigma(F)$ and $E'' \notin \sigma(F)$, thus $E \sqsupseteq_F^{\mathsf{LD}^\sigma} E''$. If $E \in \sigma(F)$, there is no set ranked strictly better than E with respect to LD^σ, hence $E \sqsupseteq_F^{\mathsf{LD}^\sigma} E''$. $\square$

A.2 TECHNICAL PROOFS OF CHAPTER 4

Proposition 2. *If extension-ranking semantics τ satisfies* respect conflicts*, then τ satisfies* cf-soundness.

Proof. Let τ be an extension-ranking semantics that satisfies *respect conflicts*, and let $F = (A, R)$ be an AF with $E \subseteq A$ and $E \in \mathtt{max}_\tau(F)$. Since $\emptyset$ is always conflict-free and we know that $E \sqsupseteq_F^\tau \emptyset$, it follows that E has to be conflict-free as well. Otherwise, *respect conflicts* would be violated. $\square$

Proposition 3. LD^σ *satisfies* σ-generalisation *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

Proof. Let $F = (A, R)$ be an AF and $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{sst}\}$. For every set $E \in \sigma(F)$, by definition, $E \sqsupset_F^{\mathsf{LD}^\sigma} E'$ for every $E' \notin \sigma(F)$. Since LD^σ produces only a binary classification, this implies that every σ-extension is among the most plausible sets, and every non-σ-extension is not among the most plausible sets. Hence, σ-*generalisation* is satisfied. $\square$

Proposition 4. *If σ is an extension semantics such that there exists an AF $F = (A, R)$ with $\sigma(F) = \emptyset$, then no extension-ranking semantics can satisfy* σ-soundness.

Proof. Let σ be an extension semantics and $F = (A, R)$ an AF such that $\sigma(F) = \emptyset$. Let τ be an extension-ranking semantics. For τ to satisfy σ-soundness, it must hold that $\texttt{max}_\tau(F) \subseteq \sigma(F)$. However, for any preorder there always exists a maximal set, meaning $\texttt{max}_\tau(F) \neq \emptyset$. Since $\sigma(F) = \emptyset$, σ-soundness is always violated. □

Proposition 5. $\mathsf{LD}^{\mathsf{cf}}$ *satisfies* respect conflicts.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ with $E \in \mathsf{cf}(F)$, and $E' \notin \mathsf{cf}(F)$. By definition, $E \sqsupset_F^{\mathsf{LD}^{\mathsf{cf}}} E'$. □

Proposition 6. LD^σ *satisfies* composition *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof. Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF. To prove the theorem for $E, E' \subseteq A_1 \cup A_2$, it must hold that if

$$\left\{ \begin{array}{l} E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{LD}^\sigma} E' \cap A_1 \\ E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{LD}^\sigma} E' \cap A_2 \end{array} \right\} \text{ then } E \sqsupseteq_F^{\mathsf{LD}^\sigma} E'.$$

We examine each semantics case by case, starting with conflict-freeness and then admissibility. Note that $\sqsupseteq^{\mathsf{LD}^\sigma}$ has only two levels, so for any set E it holds that $E \in \texttt{max}_{\mathsf{LD}^\sigma}(F)$ or $E \notin \texttt{max}_{\mathsf{LD}^\sigma}(F)$ for every AF F. So, it is sufficient to show that if $E \cap A_i$ satisfies σ in F_1 and F_2 for $i \in \{1, 2\}$, then it also satisfies σ in F and if $E \cap A_i$ violates σ in F_1 or F_2 for $i \in \{1, 2\}$ then E also violates σ in F. Since if E satisfies σ in F, then there cannot be any set S ranked strictly better than E. If $E \cap A_i$ violates σ in either F_1 or F_2 and $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{LD}^\sigma} E' \cap A_1$ or $E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{LD}^\sigma} E' \cap A_2$, then $E' \cap A_i$ violates σ in F_1 or F_2 aswell for $i \in \{1, 2\}$. So if both E and E' are violating σ in F, then neither of these sets can be ranked strictly better.

"$\sigma = \mathsf{cf}$": If $E \cap A_1$ and $E \cap A_2$ are conflict-free in F_1 and in F_2, then E is conflict-free in F, since the union of F_1 and F_2 does not introduce new conflicts.

If $E \cap A_1$ or $E \cap A_2$ is not conflict-free in F_1 or F_2, the responsible conflict $(a, b) \in R_{\{1,2\}}$ with $a, b \in E$ is also present of F. Hence, E is not conflict-free in F.

"$\sigma = \mathsf{ad}$": If $E \cap A_1$ and $E \cap A_2$ are admissible in F_1 and in F_2, then every argument in E is defended in F. Otherwise, there has to be an argument attacking an argument in E in F that is not counterattacked, however this attacker has to exists in either F_1 or F_2 as well, implying that $E \cap A_i$ can no longer be admissible in the respective F_i for $i \in \{1, 2\}$. Since, E also has to be conflict-free, E is admissible in F.

Wlog. assume $E \cap A_1$ is not admissible in F_1, then there exists an argument $a \in E$ for which there is no defender in E and since F_1 and F_2 are disjoint argument a remains not defended in F. Thus, E is not admissible in F.

"$\sigma =$ co": Assume $E \cap A_1$ and $E \cap A_2$ are complete in F_1 and F_2. If E is not complete in F there has to be an argument, which is defended by E and not contained. However, this argument must exists in either F_1 or F_2 as well, making $E \cap A_i$ no longer complete in the respective F_i for $i \in \{1, 2\}$.

Wlog. assume $E \cap A_1$ is not complete in F_1, then there exists an argument a which is defended by E and not contained in E. However, since F_1 and F_2 are disjoint this argument remains defended in F by E. Thus, E cannot be complete in F.

"$\sigma =$ gr": Assume $E \cap A_1$ and $E \cap A_2$ are grounded in F_1 and F_2, E is the least fixed point of the characteristic function in both these AFs. Since F_1 and F_2 are disjoint, every argument defended in F_1 and F_2 is also defended in F. So, E is a fixed point of the characteristic function in F. Since, the set of unattacked arguments in F is the same as the union of unattacked arguments in F_1 and F_2, E is minimal, proving that E is grounded in F.

Wlog. assume $E \cap A_1$ is not grounded in F_1, then there is a set $S \subset E$, i. e., there is an argument $a \in E$ and $a \notin S$, but $S \cup \{a\}$ defends a. This argument a is also part of F, since $F = F_1 \cup F_2$ holds as well as F_1 and F_2 are disjoint. Thus, E is not minimal in F and therefore, not grounded.

"$\sigma =$ pr": Assume $E \cap A_1$ and $E \cap A_2$ are preferred in F_1 and F_2, E is admissible in F. If E is not preferred in F, then there has to be a preferred set $S \subseteq A_1 \cup A_2$ such that $E \subset S$. Meaning there is an argument $a \in S$ and $a \notin E$, which is defended by S and E. However, a has to exits in either F_1 or F_2 and therefore $E \cap A_i$ is not maximal in that F_i for $i \in \{1, 2\}$.

Wlog. assume $E \cap A_1$ is not preferred in F_1, so there is a preferred set $S \subseteq A_1$ such that $E \subset S$. Meaning there is an argument $a \in S$ and $a \notin E$, which is defended by S and E. Since F_1 and F_2 are disjoint, a must be defended by E in F, therefore E cannot be maximal in F.

"$\sigma =$ st": Assume $E \cap A_1$ and $E \cap A_2$ are stable in F_1 and F_2, if E is not stable in F, then there has to be an argument $a \in A_1 \cup A_2$ that is not attacked by E and $a \notin E$. However, this a has to exits in F_1 or F_2 making $E \cap A_i$ not stable in that F_i for $i \in \{1, 2\}$.

Wlog. assume $E \cap A_1$ is not stable in F_1, there must be an argument $a \in A_1$ such that $a \notin E \cup E^+$. However, a has to exists in F, since $F = F_1 \cup F_2$ and F_1 and F_2 are disjoint.

"$\sigma =$ sst": Assume $E \cap A_1$ and $E \cap A_2$ are semi-stable in F_1 and F_2, E is complete in F. If E is not semi-stable in F, then there must be a semi-stable set $S \subseteq A_1 \cup A_2$ such that $E \subset S$. Meaning there is an argument $a \in S \cup S_F^+$ and $a \notin E$, which is attacked by S or in S. However, a has to exits in either F_1 or F_2 and therefore $E \cap A_i$ cannot be semi-stable in that F_i for $i \in \{1, 2\}$.

Wlog. suppose $E \cap A_1$ is not semi-stable in F_1, so there is a semi-stable set $S \subseteq A_1$ such that $E \subset S$. Meaning there is an argument $a \in S \cup S^+_{F_1}$ and $a \notin E \cup E^+_{F_1}$. Since F_1 and F_2 are disjoint, a cannot be part of $E \cup E^+_F$ in F, therefore $E \cup E^+_F$ cannot be maximal in F. $\square$

Proposition 7. LD^σ *satisfies* weak reinstatement *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$ any set. Assume $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$. We will show that $E \not\sqsupset^{\mathsf{LD}^\sigma}_F E \cup \{a\}$. Since for every set E it holds that either $E \in \mathtt{max}_{\mathsf{LD}^\sigma}(F)$ or $E \notin \mathtt{max}_{\mathsf{LD}^\sigma}(F)$ and we cannot differentiate sets more, we only need to check for $E \in \sigma(F)$ and $E \notin \sigma(F)$.

1. If $E \notin \sigma(F)$: Since $\sqsupseteq^{\mathsf{LD}^\sigma}$ has only two levels and $E \notin \mathtt{max}_{\mathsf{LD}^\sigma}(F)$, there is no set $S \subseteq A$ such that $E \sqsupset^{\mathsf{LD}^\sigma}_F S$.

2. If $E \in \sigma(F)$: Then $E \in \mathtt{max}_{\mathsf{LD}^\sigma}(F)$. For $\sigma' \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$, adding a to E does not introduce any conflict, so $E \cup \{a\}$ remains acceptable with respect to σ. If $E \in \mathsf{gr}(F)$, then $E = \mathfrak{F}_F(E)$, meaning there cannot exists an argument $a \in A$ such that $a \in \mathfrak{F}_F(E)$ and $a \notin E$. Thus, every LD^σ satisfies *weak reinstatement*. $\square$

Proposition 8. LD^σ *satisfies* addition robustness *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{gr}, \mathsf{st}\}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ with $a \in E$, and $b \in E' \setminus E$. We extend F to F' such that $F' = (A, R \cup \{(a, b)\})$. To show that LD^σ satisfies *addition robustness*, it is enough to show that if $E \in \sigma(F)$, then $E \in \sigma(F')$, and if $E' \notin \sigma(F)$, then $E' \notin \sigma(F')$. Since $\sqsupseteq^{\mathsf{LD}^\sigma}_F$ has only two levels, this implies if E satisfies σ in F', then E is ranked among the best sets with respect to $\sqsupseteq^{\mathsf{LD}^\sigma}_{F'}$ and E' cannot be ranked strictly better then E. Additionally, if $E' \notin \sigma(F')$, then E cannot be ranked strictly worse than E' with respect to $\sqsupseteq^{\mathsf{LD}^\sigma}_{F'}$.

"$\sigma = \mathsf{cf}$": Let $E \in \mathsf{cf}(F)$, then $E \in \mathsf{cf}(F')$ since no conflict are added to E. Thus, for any set $E'' \subseteq A$, $E \sqsupseteq^{\mathsf{LD}^{\mathsf{cf}}}_{F'} E''$.

Let $E' \notin \mathsf{cf}(F)$, then no conflicts are removed in F' with respect to E'. Thus, for any set $E'' \subseteq A$, $E'' \sqsupseteq^{\mathsf{LD}^{\mathsf{cf}}}_{F'} E'$. Therefore, if E is conflict-free this set stays conflict-free in F' and if a set is not conflict-free, then this set stays not conflict-free. Hence, the relationship between E and E' is unchanged with respect to $\sqsupseteq^{\mathsf{LD}^{\mathsf{cf}}}_F$.

"$\sigma = \mathsf{ad}$": Let $E \in \mathsf{ad}(F)$, E is conflict-free in F'. We need to show that every argument in E is defended in F'. Assume $a' \in E$ is not defended in F' by E, there is at least one attacker of a' that is not defeated in F' and we do not remove any attack, therefore the added attack (a, b) has to be the reason. Implying $a' = b$, which is impossible since $b \notin E$. Hence, E has to be an admissible set in F'.

Let $E' \notin \mathsf{ad}(F)$ and assume $E' \in \mathsf{ad}(F')$, there is one argument $b' \in E'$ which is not defended in F by E' but defended in F'. Let c be the attacker of b', then E' does not contain any attacker of c in F, hence the addition of (a, b) will create a new attacker for c implying $c = b$ and $a \in E'$, which implies that E' is not conflict-free in F' and therefore also not an admissible set. If E is an admissible set in F it remains an admissible set and if E' is not an admissible set E' remains not an admissible set implying that the relationship between E and E' is unchanged in F', i. e., $E \sqsupseteq_{F'}^{\mathsf{LD}^{\mathsf{ad}}} E'$.

“$\sigma = \mathsf{gr}$”: Assume $E' \in \mathsf{gr}(F')$ and $E' \notin \mathsf{gr}(F)$, since E' is not empty there must to be an unattacked argument $c \in E'$ and since $a \in b^-_{F'}$, $c \neq b$. Additionally, for every $c \neq a$, $c^+_F = c^+_{F'}$, hence every argument defended by c in F' is also defended by c in F. Since $E' \in \mathsf{gr}(F')$, every argument in E' is part of the least fixed point of $\mathfrak{F}_{F'}(\{c\})$ and everything defended by $\{c\}$ in F' is also defended by $\{c\}$ in F, therefore every argument, which is part of the least fixed point of $\mathfrak{F}_{F'}(\{c\})$ also has to be part of the least fixed of $\mathfrak{F}_F(\{c\})$, implying that $E' \in \mathsf{gr}(F)$ contradicting the assumption.

Let $E \in \mathsf{gr}(F)$, since the grounded extension is unique, it holds that $E' \notin \mathsf{gr}(F)$ and therefore $E' \notin \mathsf{gr}(F')$. Hence, it is impossible that $E' \sqsupset_{F'}^{\mathsf{LD}^{\mathsf{gr}}} E$.

“$\sigma = \mathsf{st}$”: Let $E \in \mathsf{st}(F)$, then $E \in \mathsf{ad}(F)$ and $E \in \mathsf{ad}(F')$. Additionally, no attack is removed by switching over to F', $E^+_F \subseteq E^+_{F'}$, and $E \cup E^+_F = A$. Hence, it has to hold that $E \cup E^+_{F'} = A$, implying $E \in \mathsf{st}(F')$.

Next, we show that if $E' \notin \mathsf{st}(F)$, then $E' \notin \mathsf{st}(F')$. For contradiction assume $E' \in \mathsf{st}(F')$. If $E' \notin \mathsf{cf}(F)$ or $E' \notin \mathsf{ad}(F)$, we are done. Let $E' \in \mathsf{ad}(F)$ and therefore $E' \in \mathsf{ad}(F')$. There exist an argument $b' \in E'^+_{F'}$ such that $b' \notin E'^+_F$ and additionally it holds that $E'^+_F \subset E'^+_{F'}$. Since, the only attack we add is (a, b), implying $b = b'$ and $a \in E'$. However, this also implies that E' is no longer conflict-free in F' since $b \in E'$. Therefore, b' cannot exist and therefore $E' \notin \mathsf{st}(F')$. □

To show that a pair of principles is pairwise independence, we partially introduce some new extension-ranking semantics. We start with $\mathsf{lex}(\sqsupseteq^{\mathsf{c\text{-}Conflicts}}, \sqsupseteq^{\mathsf{Kelly_{cat}}})$, which satisfies *respect conflicts* and violated *composition*.

Lemma A.1. $\mathsf{lex}(\sqsupseteq^{\mathsf{c\text{-}Conflicts}}, \sqsupseteq^{\mathsf{Kelly_{cat}}})$ *satisfies* respect conflicts *and violates* composition.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Then $\mathsf{Conflicts}_F(E) = \emptyset$ and $\mathsf{Conflicts}_F(E') \neq \emptyset$, so $E \sqsupset_F^{\mathsf{c\text{-}Conflicts}} E'$ and therefore $E \sqsupset_F^{\mathsf{lex(c\text{-}Conflicts,Kelly_{cat})}} E'$.

$\mathsf{lex}(\sqsupseteq^{\mathsf{c\text{-}Conflicts}}, \sqsupseteq^{\mathsf{Kelly_{cat}}})$ violates *composition*.

Consider $F_{51} = (\{a, b, c, d, e\}, \{(a, b), (e, c), (c, d), (c, c), (d, d)\})$ as depicted in Figure A.1. F_{51} can be partitioned into two disjoint AFs $F_{51,1} = (\{a, b\}, \{(a, b)\})$ and

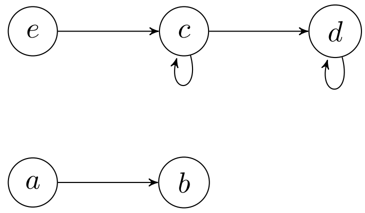

Figure A.1: AF F_{51} from Lemma A.1.

$F_{51,2} = (\{c, d, e\}, \{(e, c), (c, d), (c, c), (d, d)\})$. The corresponding argument ranking with respect to cat is:

$$e \simeq^{\mathsf{cat}}_{F_{51}} a \succ^{\mathsf{cat}}_{F_{51}} b \succ^{\mathsf{cat}}_{F_{51}} c \succ^{\mathsf{cat}}_{F_{51}} d$$

Consider sets $\{a, c\}$ and $\{b, d\}$, we get:

$$\{a\} \sqsupset^{\mathsf{lex(c\text{-}Conflicts,Kelly_{cat})}}_{F_{51,1}} \{b\} \quad \text{and} \quad \{c\} \sqsupset^{\mathsf{lex(c\text{-}Conflicts,Kelly_{cat})}}_{F_{51,2}} \{d\}$$

However, since $c \not\succ^{\mathsf{cat}}_{F_{51}} b$, we have:

$$\{a, c\} \asymp^{\mathsf{lex(c\text{-}Conflicts,Kelly_{cat})}}_{F_{51}} \{b, d\}$$

Thus, violating *composition*.

□

Next, we discuss a variant of the least-discriminating extension-ranking semantics.

Definition 97. *Let $F = (A, R)$ be an AF. Given an extension semantics σ, we define a variation of the* least-discriminating extension-ranking semantics *with respect to σ, denoted* $\mathsf{LD}^{\sigma *}$*, by:*

$$E \sqsupset^{\mathsf{LD}^{\sigma *}}_{F} E' \text{ if } E \in \sigma(F) \text{ and } E' \notin \sigma(F),$$
$$E \asymp^{\mathsf{LD}^{\sigma *}}_{F} E' \text{ if } E, E' \notin \sigma(F),$$
$$\text{and } E \equiv^{\mathsf{LD}^{\sigma *}}_{F} E', \text{ if } E, E' \in \sigma(F).$$

Lemma A.2. $\mathsf{LD}^{\sigma *}$ *satisfies* σ-generalisation *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ *and violates* weak reinstatement.

Proof. Let $F = (A, R)$ be an AF and σ an an extension semantics. Consider $E \subseteq A$ such that $E \in \sigma(F)$. Then E is ranked better than any non-extension and ranked equal to every σ-extension, so $E \in \mathtt{max}_{\mathsf{LD}^{\sigma *}}(F)$.

Consider AF F_9 from Example 30 and recalled in Figure A.2. Suppose $E = \{a\}$, then $\{a\}$ is not conflict-free, implying $\{a\} \notin \sigma(F_9)$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{gr}, \mathsf{st}, \mathsf{sst}\}$, and $b \in \mathfrak{F}_{F_9}(\{a\})$, $b \notin \{a\}$ and $b \notin (\{a\}^- \cup \{a\}^+)$. However, $\{a, b\}$ is not conflict-free, thus:

$$\{a\} \asymp^{\mathsf{LD}^{\sigma *}}_{F_9} \{a, b\}$$

Therefore, $\mathsf{LD}^{\sigma *}$ violates *weak reinstatement*. □

Figure A.2: AF F_9 from Example 30.

The extension-ranking semantics, which states that every pair of sets of arguments are incomparable satisfies *decomposition* and violates *weak reinstatement*.

Definition 98. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. The* incomparable *extension-ranking semantics $\sqsupseteq^{\mathsf{incomparable}}$ is defined via:*

$$E \equiv_F^{\mathsf{incomparable}} E' \textit{ if and only if } E = E'$$
$$E \asymp_F^{\mathsf{incomparable}} E' \textit{ otherwise}$$

Lemma A.3. $\sqsupseteq^{\mathsf{incomparable}}$ *satisfies* decomposition *and violates* weak reinstatement.

Proof. Let $F = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and $E, E' \subseteq A_1 \cup A_2$. By definition, if:

$$E \equiv_F^{\mathsf{incomparable}} E'$$

Then:

$$E \cap A_1 \equiv_{F_1}^{\mathsf{incomparable}} E' \cap A_1 \quad \text{and} \quad E \cap A_2 \equiv_{F_2}^{\mathsf{incomparable}} E' \cap A_2$$

Similarly, if:

$$E \asymp_F^{\mathsf{incomparable}} E'$$

Then:

$$E \cap A_1 \asymp_{F_1}^{\mathsf{incomparable}} E' \cap A_1 \quad \text{and} \quad E \cap A_2 \asymp_{F_2}^{\mathsf{incomparable}} E' \cap A_2$$

Next assume $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$. Then $E \neq E \cup \{a\}$, hence:

$$E \asymp_F^{\mathsf{incomparable}} E \cup \{a\}$$

Thus, *weak reinstatement* is violated. □

Next, we discuss an argument-ranking semantics inspired by the principle *CP*.

Definition 99. *Let $F = (A, R)$ be an AF and $a, b \in A$. We define the* cardinality-based *argument-ranking semantics* Cb *as follows:*

$$a \succeq_F^{\mathsf{Cb}} b \textit{ if and only if } |a_F^-| \leq |b_F^-|$$

In other words, an argument a is stronger than argument b if a has fewer direct attackers than b. It is straightforward to see that Cb satisfies *CP* and *Abs*.

Lemma A.4. $\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}$ *satisfies* addition robustness.

Proof. Let $F = (A, R)$ be an AF, $\succeq_F^{\mathsf{Cb}}$ the argument ranking with respect to Cb, and $E, E' \subseteq A$ such that $E \sqsupseteq_F^{\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}} E'$. Consider AF $F' = (A, R \cup \{(a, b)\}$ such that $a \in E$, $b \notin E$, and $b \in E'$. For every argument $c \in A \setminus \{b\}$, $c_F^- = c_{F'}^-$, and for b, $b_{F'}^- = b_F^- \cup \{a\}$.

If $b \succ_F^{\mathsf{Cb}} c$ then $|b_F^-| < |c_F^-|$ and therefore we have either $b \succ_{F'}^{\mathsf{Cb}} c$ or $b \simeq_{F'}^{\mathsf{Cb}} c$. The first case implies that $sv_F^{\mathsf{Cb}}(b) = sv_{F'}^{\mathsf{Cb}}(b)$ and the second case implies $sv_F^{\mathsf{Cb}}(b) < sv_{F'}^{\mathsf{Cb}}(b)$.

If $b \simeq_F^{\mathsf{Cb}} c$ then $c \succ_{F'}^{\mathsf{Cb}} b$ and this implies $sv_F^{\mathsf{Cb}}(b) < sv_{F'}^{\mathsf{Cb}}(b)$ and $sv_F^{\mathsf{Cb}}(c) > sv_{F'}^{\mathsf{Cb}}(c)$.

Theses two cases show that for E, $\sum_{c \in E} sv_F^{\mathsf{Cb}}(c) \geq \sum_{c \in E} sv_{F'}^{\mathsf{Cb}}(c)$. Every time the value $sv_{F'}^{\mathsf{Cb}}(c)$ is lower than in F, $sv_{F'}^{\mathsf{Cb}}(b)$ increases and ensuring $\sum_{d \in E'} sv_F^{\mathsf{Cb}}(d) \leq \sum_{d \in E'} sv_{F'}^{\mathsf{Cb}}(d)$. Therefore:

$$E \sqsupseteq_{F'}^{\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}} E'$$

□

To show that *composition* is violated by $\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}$, consider Example 107. Next, we show that $\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}$ violates *weak reinstatement*.

Example 157. *Consider AF F_5 from Example 22 and set $\{h\}$. Then argument j is defended by $\{h\}$, but j has one attacker implying:*

$$\{h\} \sqsupset_{F_5}^{\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}} \{h, j\}$$

Thus, weak reinstatement *is violated.*

The extension-ranking semantics $\mathsf{lex}(\sqsupseteq^{\mathsf{Conflicts}}, \subseteq)$ satisfies *respect conflicts* and violates *weak reinstatement*.

Lemma A.5. $\mathsf{lex}(\sqsupseteq^{\mathsf{Conflicts}}, \subseteq)$ *satisfies* respect conflicts *and violates* weak reinstatement.

Proof. Let $F = (A, R)$ be an AF.

Consider $E, E' \subseteq A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Then $\mathsf{Conflicts}_F(E) = \emptyset$ and $\mathsf{Conflicts}_F(E') \neq \emptyset$, implying:

$$E \sqsupset_F^{\mathsf{Conflicts}} E'$$

Therefore:

$$E \sqsupset_F^{\mathsf{lex}(\mathsf{Conflicts},\subseteq)} E'$$

Thus, *respect conflicts* is satisfied.

Consider $E \subseteq A$ and $a \in A$ such that $a \notin E_F^+ \cup E_F^-$ with $a \in \mathfrak{F}_F(E)$. We know $\mathsf{Conflicts}_F(E) = \mathsf{Conflicts}_F(E \cup \{a\})$, therefore:

$$E \equiv_F^{\mathsf{Conflicts}} E \cup \{a\}$$

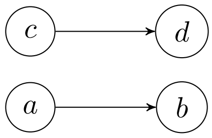

Figure A.3: AF F_8 from Example 29.

However, E is a subset of $E \cup \{a\}$, meaning:

$$E \sqsupset_F^{\mathsf{lex}(\sqsupseteq^{\mathsf{Conflicts}}, \subseteq)} E \cup \{a\}$$

Therefore, *weak reinstatement* is violated. $\square$

Next, we introduce an extension-ranking semantics that satisfies *decomposition* and violates *composition*.

Definition 100. *Let $F = (A, R)$ be an AF and $a, b \in A$ be two specific arguments. We define the* exclusive-or extension-ranking semantics $\mathsf{XOR}_{a,b}$ *as follows:*

$$\textit{If } a, b \in E \textit{ or } a, b \in E' \textit{ then } E \asymp_F^{\mathsf{XOR}_{a,b}} E',$$
$$\textit{Otherwise, } E \equiv_F^{\mathsf{XOR}_{a,b}} E'$$

Lemma A.6. $\mathsf{XOR}_{a,b}$ *satisfies* decomposition *for any two arguments a, b and violates* composition*.*

Proof. Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and two arguments $a, b \in A_1 \cup A_2$. Assume $E \sqsupset_F^{\mathsf{XOR}_{a,b}} E'$. This means that $a, b \notin E$ and $a, b \notin E'$. This also entails, $a, b \notin E \cap A_1$, $a, b \notin E \cap A_2$, $a, b \notin E' \cap A_1$, and $a, b \notin E' \cap A_2$. Hence:

$$E \cap A_1 \sqsupset_{F_1}^{\mathsf{XOR}_{a,b}} E' \cap A_1 \quad \text{and} \quad E \cap A_2 \sqsupset_{F_1}^{\mathsf{XOR}_{a,b}} E' \cap A_2$$

Showing that *decomposition* is satisfied.

Consider F_8 from Example 29 recalled in Figure A.3. For $\mathsf{XOR}_{a,c}$ and sets $\{a, c\}$ and $\{b, d\}$ we get:

$$\{a\} \equiv_{F_{8,1}}^{\mathsf{XOR}_{a,c}} \{b\} \quad \text{and} \quad \{c\} \equiv_{F_{8,2}}^{\mathsf{XOR}_{a,c}} \{d\}$$

However, we get:

$$\{a, c\} \asymp_{F_8}^{\mathsf{XOR}_{a,c}} \{b, d\}$$

Thus, *composition* is violated. A similar example can be found for any two arguments. $\square$

Proposition 9. *The following holds:*

1. *The principle* respect conflicts *implies* cf-soundness.

2. *The principle* strong reinstatement *implies* weak reinstatement.
3. *The principles* ad-generalisation *and* strong reinstatement *are incompatible.*
4. *The principles* σ-generalisation *for* $\sigma \in \{\mathsf{co}, \mathsf{pr}, \mathsf{sst}\}$ *and* addition robustness *are incompatible.*
5. *For the remaining pairs of principles* $p_1, p_2 \in \{\sigma$-generalisation, respect conflicts, composition, decomposition, weak reinstatement, strong reinstatement, addition robustness$\}$ *where* $p_1 \neq p_2$, p_1 *and* p_2 *are pairwise independent.*

Proof. **to 1** Proposition 2 shows that *respect conflicts* implies cf-*soundness*.

to 2 Let τ be an extension-ranking semantics that satisfies *strong reinstatement*. For any $F = (A, R)$ with $E \subseteq A$, if $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E_F^- \cup E_F^+)$, it follows that $E \cup \{a\} \sqsupset_F^\tau E$. Therefore, $E \cup \{a\} \sqsupseteq_F^\tau E$ also holds.

to 3 Example 32 shows that ad-*generalisation* and *strong reinstatement* are incompatible.

to 4 Example 31 shows that σ-*generalisation* for $\sigma \in \{\mathsf{co}, \mathsf{pr}, \mathsf{sst}\}$ and *addition robustness* are incompatible.

to 5 To show the pairwise independence of the remaining pairs of principles, for every pair p_1 and p_2, we present:

- One extension-ranking semantics that satisfies p_1 and p_2.
- One extension-ranking semantics that satisfies p_1 but violates p_2.
- One extension-ranking semantics that violates p_1 but satisfies p_2.

(σ***-generalisation***, ***respect conflicts***)**:**

- r-σ satisfies both principles (see Table 6.1),
- LD^σ satisfies σ-*generalisation* and violates *respect conflicts* (see Table 4.1),
- $\sqsupseteq^{\mathsf{Conflicts}}$ violates σ-*generalisation* and satisfies *respect conflicts* (see Table 5.1).

(σ***-generalisation***, ***composition***)**:**

- r-σ satisfies both principles (see Table 6.1),
- $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ ($\sigma = \mathsf{ad}$) satisfies σ-*generalisation* and violates *composition* (see Table 6.3),
- $\sqsupseteq^{\mathsf{Conflicts}}$ violates σ-*generalisation* and satisfies *composition* (see Table 5.1).

(σ***-generalisation***, ***decomposition***)**:**

- r-σ satisfies both principles (see Table 6.1),

- LD^{σ} satisfies *σ-generalisation* and violates *decomposition* (see Table 4.1),
- $\sqsupseteq^{\mathsf{Conflicts}}$ violates *σ-generalisation* and satisfies *decomposition* (see Table 5.1).

(***σ-generalisation***, ***weak reinstatement***):

- $\mathsf{r}\text{-}\sigma$ satisfies both principles (see Table 6.1),
- $\mathsf{LD}^{\sigma *}$ satisfies *σ-generalisation* and violates *weak reinstatement* (see Lemma A.2),
- $\sqsupseteq^{\mathsf{Conflicts}}$ violates *σ-generalisation* and satisfies *weak reinstatement* (see Table 5.1).

(***σ-generalisation***, ***strong reinstatement***):

- $\mathsf{r}\text{-}\sigma$ ($\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$) satisfies both principles (see Table 6.1),
- LD^{σ} satisfies *σ-generalisation* and violates *strong reinstatement* (see Table 4.1),
- $\sqsupseteq^{\mathsf{Condef}}$ violates *σ-generalisation* and satisfies *strong reinstatement* (see Table 5.1).

(ad***-generalisation***, ***addition robustness***):

- $\mathsf{LD}^{\mathsf{ad}}$ satisfies both principles (see Table 4.1),
- $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ satisfies ad-*generalisation* and violates *addition robustness* (see Table 6.3),
- $\sqsupseteq^{\mathsf{Conflicts}}$ violates ad-*generalisation* and satisfies *addition robustness* (see Table 5.1).

(***respect conflicts***, ***composition***):

- $\mathsf{r}\text{-}\sigma$ satisfies both principles (see Table 6.1),
- $\mathsf{lex}(\sqsupseteq^{\mathsf{c\text{-}Conflicts}}, \sqsupseteq^{\mathsf{Kelly_{cat}}})$ satisfies *respect conflicts* and violates *composition* (see Lemma A.1),
- $\sqsupseteq^{\mathsf{Undef}}$ violates *respect conflicts* and satisfies *composition* (see Table 5.1).

(***respect conflicts***, ***decomposition***):

- $\mathsf{r}\text{-}\sigma$ satisfies both principles (see Table 6.1),
- $\mathsf{LD}^{\mathsf{Conflicts}}$ satisfies *respect conflicts* and violates *decomposition* (see Table 4.1),
- $\sqsupseteq^{\mathsf{Undef}}$ violates *respect conflicts* and satisfies *decomposition* (see Table 5.1).

(***respect conflicts***, ***weak reinstatement***):

- $\mathsf{r}\text{-}\sigma$ satisfies both principles (see Table 6.1),
- $\mathsf{lex}(\sqsupseteq^{\mathsf{Conflicts}}, \subseteq)$ satisfies *respect conflicts* and violates *weak reinstatement* (see Lemma A.5),

- $\sqsupseteq^{\mathsf{Undef}}$ violates *respect conflicts* and satisfies *weak reinstatement* (see Table 5.1).

(***respect conflicts***, ***strong reinstatement***):

- r-co satisfies both principles (see Table 6.1),
- $\sqsupseteq^{\mathsf{Conflicts}}$ satisfies *respect conflicts* and violates *strong reinstatement* (see Table 5.1),
- $\sqsupseteq^{\mathsf{Condef}}$ violates *respect conflicts* and satisfies *strong reinstatement* (see Table 5.1).

(***respect conflicts***, ***addition robustness***):

- r-ad satisfies both principles (see Table 6.1),
- r-co satisfies *respect conflicts* and violates *addition robustness* (see Table 6.1),
- $\sqsupseteq^{\mathsf{Unatt}}$ violates *respect conflicts* and satisfies *addition robustness* (see Table 5.1).

(***composition***, ***decomposition***):

- r-σ satisfies both principles (see Table 6.1),
- $\sqsupseteq^{\mathsf{Conflicts}}$ satisfies *composition* and violates *decomposition* (see Table 5.1),
- $\mathsf{XOR}_{a,b}$ violates *composition* and satisfies *decomposition* (see Lemma A.6).

(***composition***, ***weak reinstatement***):

- r-σ satisfies both principles (see Table 6.1),
- $\mathsf{GCB}_{Cat,\mathsf{leximax}}$ satisfies *composition* and violates *weak reinstatement* (see Table 8.3),
- cope($\sqsupseteq^{\mathsf{Conflicts}}$, $\sqsupseteq^{\mathsf{Undef}}$) violates *composition* and satisfies *weak reinstatement* (see Table 6.3).

(***composition***, ***strong reinstatement***):

- r-co satisfies both principles (see Table 6.1),
- r-ad satisfies *composition* and violates *weak reinstatement* (see Table 6.1),
- cope($\sqsupseteq^{\mathsf{nonatt}}$) violates *composition* and satisfies *strong reinstatement* (see Table 6.3).

(***composition***, ***addition robustness***):

- r-ad satisfies both principles (see Table 6.1),
- r-co satisfies *composition* and violates *addition robustness* (see Table 6.1),
- $\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}$ violates *composition* and satisfies *addition robustness* (see Lemma A.4).

(***decomposition***, ***weak reinstatement***):

- r-σ satisfies both principles (see Table 6.1),
- $\sqsupseteq^{\mathsf{incomparable}}$ satisfies *decomposition* and violates *weak reinstatement* (see Lemma A.3),
- LD^{σ} violates *decomposition* and satisfies *weak reinstatement* (see Table 4.1).

(***decomposition***, ***strong reinstatement***):

- r-co satisfies both principles (see Table 6.1),
- r-ad satisfies *decomposition* and violates *weak reinstatement* (see Table 6.1),
- cope($\sqsupseteq^{\mathsf{nonatt}}$) violates *decomposition* and satisfies *strong reinstatement* (see Table 6.3).

(***decomposition***, ***addition robustness***):

- r-ad satisfies both principles (see Table 6.1),
- r-co satisfies *decomposition* and violates *addition robustness* (see Table 6.1),
- $\mathsf{LD}^{\mathsf{ad}}$ violates *decomposition* and satisfies *addition robustness* (see Table 4.1).

(***weak reinstatement***, ***strong reinstatement***):

- r-co satisfies both principles (see Table 6.1),
- r-ad satisfies *weak reinstatement* and violates *strong reinstatement* (see Table 6.1).

(***weak reinstatement***, ***addition robustness***):

- r-ad satisfies both principles (see Table 6.1),
- r-co satisfies *weak reinstatement* and violates *addition robustness* (see Table 6.1),
- $\mathsf{GCB}_{sv^{\mathsf{Cb}},\mathsf{sum}}$ violates *weak reinstatement* and satisfies *addition robustness* (see Lemma A.4).

(***strong reinstatement***, ***addition robustness***):

- r-co satisfies *strong reinstatement* and violates *addition robustness* (see Table 6.1),
- $\mathsf{LD}^{\mathsf{ad}}$ violates *strong reinstatement* and satisfies *addition robustness* (see Table 4.1). □

A.3 TECHNICAL PROOFS OF CHAPTER 5

Proposition 10. *Let $F = (A, R)$ be an AF and $E \subseteq A$, then exists a $k \in \mathbb{N}$ with*

$$E \subseteq \mathfrak{F}^*_{1,F}(E) \subseteq \mathfrak{F}^*_{2,F}(E) \subseteq \cdots \subseteq \mathfrak{F}^*_{k-1,F}(E) \subseteq \mathfrak{F}^*_{k,F}(E) = \mathfrak{F}^*_{k+1,F}(E) = \ldots$$

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. We show that $\mathfrak{F}^*_{i,F}(E) \subseteq \mathfrak{F}^*_{i+1,F}(E)$.

Observe that $\mathfrak{F}^*_{i,F}(E)$ is formed by adding arguments from $(\mathfrak{F}_F(\mathfrak{F}^*_{i-1,F}(E)) \setminus E^-)$, without removing any arguments:

$$\mathfrak{F}^*_{i,F}(E) = \mathfrak{F}^*_{i-1,F}(E) \cup (\mathfrak{F}_F(\mathfrak{F}^*_{i-1,F}(E)) \setminus E^-)$$

Therefore:

$$\begin{aligned}
&\mathfrak{F}^*_{i-1,F}(E) \cup (\mathfrak{F}_F(\mathfrak{F}^*_{i-1,F}(E)) \setminus E^-) \subseteq \\
&(\mathfrak{F}^*_{i-1,F}(E) \cup (\mathfrak{F}_F(\mathfrak{F}^*_{i-1,F}(E)) \setminus E^-)) \\
&\quad \cup (\mathfrak{F}_F(\mathfrak{F}^*_{i-1,F}(E) \cup (\mathfrak{F}_F(\mathfrak{F}^*_{i-1,F}(E)) \setminus E^-)) \setminus E^-)
\end{aligned}$$

This is equivalent to:

$$\mathfrak{F}^*_{i,F}(E) \subseteq \mathfrak{F}^*_{i+1,F}(E)$$

Since $\mathfrak{F}^*_{i,F}(E) \subseteq \mathfrak{F}^*_{i+1,F}(E)$ holds for every i, there must exist a smallest k such that $\mathfrak{F}^*_{k,F}(E) = \mathfrak{F}^*_{k+1,F}(E)$, because the number of arguments is finite. Therefore, $\mathfrak{F}^*_{k,F}(E) = \mathfrak{F}^*_{k',F}(E)$ for all $k' > k$. □

Proposition 11. *Let $F = (A, R)$ be an AF, $E \subseteq A$ such that $E \in \mathsf{ad}(F)$, then $\mathfrak{F}_F(E) = \mathfrak{F}^*_F(E)$.*

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$ such that $E \in \mathsf{ad}(F)$. Applying $\mathfrak{F}^*$ to E, we get: $\mathfrak{F}^*_{1,F}(E) = E$ and $\mathfrak{F}^*_{2,F}(E) = E \cup (\mathfrak{F}_F(E) \setminus E^-)$. $\mathfrak{F}_F(E)$ is defined such that this function will not return any attacker of E if E is admissible, so we do not need to remove any attacker, thus, $\mathfrak{F}_F(E) \setminus E^- = \mathfrak{F}_F(E)$. The *Fundamental Lemma* (Dung, 1995) then implies $E \subseteq \mathfrak{F}_F(E)$ if E is admissible, $E \cup \mathfrak{F}_F(E) = \mathfrak{F}_F(E)$. Therefore, $\mathfrak{F}^*_{2,F}(E) = \mathfrak{F}_F(E)$. We can use the same reasoning for any $i > 2$ and entailing that in every step $\mathfrak{F}^*_F(E)$ return the same arguments as $\mathfrak{F}_F(E)$. □

Proposition 12. *Let $F = (A, R)$ be an AF and $E = \mathsf{gr}(F)$, then E is the least fixed point of $\mathfrak{F}^*_F$.*

Proof. Let $F = (A, R)$ be an AF and $E = \mathsf{gr}(F)$. To prove that E is the least fixed point of $\mathfrak{F}^*_F$, we modify the algorithm to calculate the grounded extension (for details see Baroni et al. (2018)). This algorithm starts with the empty set, $\mathfrak{F}_F(\emptyset)$, and add in every step all defended arguments into that set until a fixed point is reached. Since $\emptyset \in \mathsf{ad}(F)$, $\mathfrak{F}_F(\emptyset) = \mathfrak{F}^*_F(\emptyset)$ like shown in Proposition 11 and if $\mathfrak{F}_F(\emptyset)$ reaches a fixed point, then $\mathfrak{F}^*_F(\emptyset)$ reaches a fixed point. Since the grounded extension E is an admissible set, $\mathfrak{F}_F(E) = \mathfrak{F}^*_F(E)$. This fixed point of $\mathfrak{F}^*_F(\emptyset)$ coincided with the grounded extension E and therefore E is the least fixed point of $\mathfrak{F}^*_F$. □

Proposition 13. *Let $F = (A, R)$ be an AF and $E \in \mathsf{co}(F)$, then $\mathfrak{F}^*_F(E) = E$.*

Proof. Let $F = (A, R)$ be an AF and $E \in \mathsf{co}(F)$, by definition of the complete semantics, $E \in \mathsf{ad}(F)$ and therefore $\mathfrak{F}_F(E) = \mathfrak{F}^*_F(E)$. Since, E already contains every argument it defends, no more arguments will be added when applying the characteristic function, $\mathfrak{F}_F(E) \subseteq E$. The other direction is shown by Proposition 11. Thus, $\mathfrak{F}^*_F(E) = E$. □

Proposition 14. *The base relation $\sqsupseteq^\tau_F$ is reflexive and transitive for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.*

Proof. Consider AF $F = (A, R)$, $E \subseteq A$, and $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.

reflexive Let $\mathcal{E} = \tau_F(E)$ and $E' = E$, then $\tau_F(E') = \mathcal{E}$ and therefore $E \equiv^\tau_F E'$.

transitive Let $E', E'' \subseteq A$ be two sets of arguments such that $E \sqsupseteq^\tau_F E'$ and $E' \sqsupseteq^\tau_F E''$. Then $\tau_F(E) \subseteq \tau_F(E')$ and $\tau_F(E') \subseteq \tau_F(E'')$, therefore $E \sqsupseteq^\tau_F E''$. □

Proposition 15. *Let $F = (A, R)$ be an AF, then $\mathsf{cf}(F) = \mathtt{max}_{\sqsupseteq^{\mathsf{Conflicts}}}(F)$.*

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$ such that $E \in \mathsf{cf}(F)$. E is conflict-free and therefore there is no pair $a, b \in E$ such that $(a, b) \in R$, thus $\mathsf{Conflicts}_F(E) = \emptyset$ and $E \in \mathtt{max}_{\sqsupseteq^{\mathsf{Conflicts}}}(F)$.

Consider $E' \subseteq A$ such that $E' \notin \mathsf{cf}(F)$, then there is one pair $a, b \in E'$ such that $(a, b) \in R$, thus $\mathsf{Conflicts}_F(E) \neq \emptyset$ and $E' \notin \mathtt{max}_{\sqsupseteq^{\mathsf{Conflicts}}}(F)$. □

Lemma 2. *For $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, if AFs $F_1 = (A_1, R_1)$, $F_2 = (A_2, R_2)$ and $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ then $\tau_{F_1}(E \cap A_1) \cup \tau_{F_2}(E \cap A_2) = \tau_F(E)$ for every $E \subseteq A_1 \cup A_2$.*

Proof. We prove it for $\tau = \mathsf{Undef}$ (others are similar). Suppose $F_1 = (A_1, R_1)$, $F_2 = (A_2, R_2)$ and $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$. We prove that $\mathsf{Undef}_{F_1}(E \cap A_1) \cup \mathsf{Undef}_{F_2}(E \cap A_2) = \mathsf{Undef}_F(E)$.

"$\subseteq$": Suppose w.l.o.g. that argument $a \in \mathsf{Undef}_{F_1}(E \cap A_1)$. Then $a \in E \cap A_1$ and $a \notin \mathfrak{F}_{F_1}(E \cap A_1)$. Therefore, $a \in E$ and since F_1 and F_2 are disjoint, $a \notin \mathfrak{F}_F(E)$. Implying $a \in \mathsf{Undef}_F(E)$.

"$\supseteq$": Suppose $a \in \mathsf{Undef}_F(E)$. Furthermore, suppose w.l.o.g. that $a \in F_1$. Then $a \in E$ and $a \notin \mathfrak{F}_F(E)$. Hence, there is an $b \in A_1 \cup A_2$ such that b attacks a and b is not attacked by E. It then follows that $b \in A_1$. Thus, $a \notin \mathfrak{F}_{F_1}(E \cap A_1)$. Implying, $a \in \mathsf{Undef}_{F_1}(E \cap A_1)$. □

Proposition 16. *The base relation $\sqsupseteq^\tau_F$ satisfies* composition *and* decomposition *for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.*

Proof. Let $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, $F_1 = (A_1, R_1)$, $F_2 = (A_2, R_2)$, and $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$.

"Composition": Suppose $E \cap A_1 \sqsupseteq^{\tau}_{F_1} E' \cap A_1$ and $E \cap A_2 \sqsupseteq^{\tau}_{F_1} E' \cap A_2$. Then $\tau_{F_1}(E \cap A_1) \subseteq \tau_{F_1}(E' \cap A_1)$ and $\tau_{F_2}(E \cap A_2) \subseteq \tau_{F_2}(E' \cap A_2)$. Lemma 2 then implies that $\tau_F(E) \subseteq \tau_F(E')$. Thus, $E \sqsupseteq^{\tau}_F E'$.

"Decomposition": Suppose $E \sqsupseteq^{\tau}_F E'$. Then $\tau_F(E) \subseteq \tau_F(E')$. Lemma 2 then implies that $\tau_{F_1}(E \cap A_1) \subseteq \tau_{F_1}(E' \cap A_1)$ and $\tau_{F_2}(E \cap A_2) \subseteq \tau_{F_2}(E' \cap A_2)$. Thus, $E \cap A_1 \sqsupseteq^{\tau}_{F_1} E' \cap A_1$ and $E \cap A_2 \sqsupseteq^{\tau}_{F_2} E' \cap A_2$. □

Proposition 17. $\sqsupseteq^{\tau}$ *satisfies* strong reinstatement *for* $\tau \in \{\mathsf{Condef}, \mathsf{Unatt}\}$ *and* weak reinstatement *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}\}$.

Proof. Let $F = (A, R)$ be an AF, $E \subseteq A$, and $a \in A$ such that $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin E^+_F \cup E^-_F$.

"$\tau = \mathsf{Conflicts}$": Since $a \notin E^+_F \cup E^-_F$ and $a \in \mathfrak{F}_F(E)$, $\mathsf{Conflicts}_F(E) = \mathsf{Conflicts}_F(E \cup \{a\})$, thus $E \cup \{a\} \equiv^{\mathsf{Conflicts}}_F E$.

"$\tau = \mathsf{Undef}$": Since $a \in \mathfrak{F}_F(E)$, $\mathsf{Undef}_F(E \cup \{a\}) \subseteq \mathsf{Undef}_F(E)$, implying $E \cup \{a\} \sqsupseteq^{\mathsf{Undef}}_F E$. However, if $a^+_F = \emptyset$, then $\mathsf{Undef}_F(E \cup \{a\}) = \mathsf{Undef}_F(E)$ and therefore $E \cup \{a\} \equiv^{\mathsf{Undef}}_F E$.

"$\tau = \mathsf{Condef}$": Since $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin E^+_F \cup E^-_F$, $a \in \mathsf{Condef}_F(E)$. We have $\mathfrak{F}^*_{2,F}(E) = E \cup (\mathfrak{F}_F(E) \setminus E^-_F)$ and $a \in (\mathfrak{F}_F(E) \setminus E^-_F)$, implying $\mathfrak{F}^*_{2,F}(E) \subseteq \mathfrak{F}^*_{2,F}(E \cup \{a\})$. Then Proposition 10 entails $\mathfrak{F}^*_F(E) = \mathfrak{F}^*_F(E \cup \{a\})$, thus $\mathsf{Condef}(E \cup \{a\}) \subset \mathsf{Condef}(E)$ and $E \cup \{a\} \sqsupset^{\mathsf{Condef}}_F E$.

"$\tau = \mathsf{Unatt}$": Since $a \notin E \cup E^+_F$, $a \in \mathsf{Unatt}(E)$ and adding a does not remove any argument from E^+_F, so $E^+_F \subseteq (E \cup \{a\})^+_F$. Thus, $\mathsf{Unatt}(E \cup \{a\}) \subset \mathsf{Unatt}(E)$ and $E \cup \{a\} \sqsupset^{\mathsf{Unatt}}_F E$. □

Proposition 18. $\sqsupseteq^{\tau}$ *satisfies* addition robustness *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Unatt}\}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ such that $E \sqsupseteq^{\tau}_F E'$ with respect to $\tau \in \{\mathsf{Conflicts}, \mathsf{Unatt}\}$. Suppose AF $F' = (A, R \cup \{(a, b)\}$ such that $a \in E$, $b \notin E$, and $b \in E'$.

"$\tau = \mathsf{Conflicts}$": Since $b \notin E$, $\mathsf{Conflicts}_F(E) = \mathsf{Conflicts}_{F'}(E)$ and $\mathsf{Conflicts}_F(E') \subseteq \mathsf{Conflicts}_{F'}(E')$, if $E \sqsupseteq^{\mathsf{Conflicts}}_F E'$ then $E \sqsupseteq^{\mathsf{Conflicts}}_{F'} E'$.

"$\tau = \mathsf{Unatt}$": For every attack r in R, $r \in R \cup \{(a, b)\}$. Hence, for every argument $c \in A, c \neq a$, $c^+_F = c^+_{F'}$. Thus, $\mathsf{Unatt}_F(E) \supseteq \mathsf{Unatt}_{F'}(E)$ and $\mathsf{Unatt}_F(E') = \mathsf{Unatt}_{F'}(E')$, therefore if $E \sqsupseteq^{\mathsf{Unatt}}_F E'$ then $E \sqsupseteq^{\mathsf{Unatt}}_{F'} E'$. □

Proposition 19. *For AF* $F = (A, R)$, $E, E' \subseteq A$, *and* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, *it holds that if* $E \sqsupseteq^{\tau}_F E'$ *then* $E \sqsupseteq^{\mathsf{c}\text{-}\tau}_F E'$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. Consider base relations $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ and $E \sqsupseteq^{\tau}_F E'$. Then $\tau_F(E) \subseteq \tau_F(E')$, this also implies $|\tau_F(E)| \leq |\tau_F(E')|$, thus $E \sqsupseteq^{\mathsf{c}\text{-}\tau}_F E'$. □

Proposition 20. $\sqsupseteq^{\text{c-Conflicts}}$ *satisfies* respect conflicts.

Proof. Let $F = (A, R)$ be an AF and $E, E' \in A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Since $E \in \mathsf{cf}(F)$, $\mathsf{Conflicts}_F(E) = \emptyset$ and therefore $|\mathsf{Conflicts}_F(E)| = 0$. For E', $\mathsf{Conflicts}_F(E') \neq \emptyset$, hence $|\mathsf{Conflicts}_F(E')| > 0$ and $E \sqsupset_F^{\text{c-Conflicts}} E'$. □

Lemma 3. *For* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$*, if* $F_1 = (A_1, R_1)$*,* $F_2 = (A_2, R_2)$ *and* $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ *with* $A_1 \cap A_2 = \emptyset$ *then* $|\tau_{F_1}(E \cap A_1)| + |\tau_{F_2}(E \cap A_2)| = |\tau_F(E)|$ *for every* $E \subseteq A_1 \cup A_2$*.*

Proof. Suppose $F_1 = (A_1, R_1)$, $F_2 = (A_2, R_2)$ and $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$. Because of Lemma 2, $\tau_{F_1}(E \cap A_1) \cup \tau_{F_2}(E \cap A_2) = \tau_F(E)$. Every element in $\tau_{F_1}(E \cap A_1) \cup \tau_{F_2}(E \cap A_2)$ is in $\tau_F(E)$ and the other way around. Since $A_1 \cap A_2 = \emptyset$, $\tau_{F_1}(E \cap A_1)$ and $\tau_{F_2}(E \cap A_2)$ are disjoint and therefore every element of $\tau_F(E)$ can only be in $\tau_{F_1}(E \cap A_1)$ or $\tau_{F_2}(E \cap A_2)$. Thus, $|\mathsf{Conflicts}_F(E')| > 0$ and $E \sqsupset_F^{\text{c-Conflicts}} E'$. □

Proposition 21. $\sqsupseteq^{\text{c-}\tau}$ *satisfies* composition *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$.

Proof. Let $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, $F_1 = (A_1, R_1)$, $F_2 = (A_2, R_2)$ and $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$.

Suppose $E \cap A_1 \sqsupseteq_{F_1}^{\text{c-}\tau} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\text{c-}\tau} E' \cap A_2$. Then $|\tau_{F_1}(E \cap A_1)| \leq |\tau_{F_1}(E' \cap A_1)|$ and $|\tau_{F_2}(E \cap A_2)| \leq |\tau_{F_2}(E' \cap A_2)|$. Lemma 3 then implies $|\tau_F(E)| \leq |\tau_F(E')|$ and therefore $E \sqsupseteq_F^{\text{c-}\tau} E'$. □

Proposition 22. $\sqsupseteq^{\text{c-}\tau}$ *satisfies* strong reinstatement *for* $\tau \in \{\mathsf{Condef}, \mathsf{Unatt}\}$*, and* weak reinstatement *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}\}$.

Proof. Let $F = (A, R)$ be an AF, $E \subseteq A$ and $a \in A$ such that $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin E_F^+ \cup E_F^-$.

"$\tau = \mathsf{Conflicts}$": Since $a \notin E_F^+ \cup E_F^-$ and $a \in \mathfrak{F}_F(E)$, $|\mathsf{Conflicts}_F(E)| = |\mathsf{Conflicts}_F(E \cup \{a\})|$, thus $E \cup \{a\} \equiv_F^{\text{c-Conflicts}} E$.

"$\tau = \mathsf{Undef}$": Since $a \in \mathfrak{F}_F(E)$, $|\mathsf{Undef}_F(E \cup \{a\})| \leq |\mathsf{Undef}_F(E)|$, implying $E \cup \{a\} \sqsupseteq_F^{\text{c-Undef}} E$. However, if $a_F^+ = \emptyset$, then $|\mathsf{Undef}_F(E \cup \{a\})| = |\mathsf{Undef}_F(E)|$ and $E \cup \{a\} \equiv_F^{\text{c-Undef}} E$.

"$\tau = \mathsf{Condef}$": Since $a \in \mathfrak{F}_F(E)$ and $a \notin E \cup E_F^+ \cup E_F^-$, $a \in \mathsf{Condef}_F(E)$. We have $\mathfrak{F}_{2,F}^*(E) = E \cup (\mathfrak{F}_F(E) \setminus E_F^-)$ and $a \in (\mathfrak{F}_F(E) \setminus E_F^-)$, implying $\mathfrak{F}_{2,F}^*(E) \subseteq \mathfrak{F}_{2,F}^*(E \cup \{a\})$. Then Proposition 10 entails $\mathfrak{F}_F^*(E) = \mathfrak{F}_F^*(E \cup \{a\})$ and $|\mathsf{Condef}_F(E \cup \{a\})| < |\mathsf{Condef}_F(E)|$. Thus, $E \cup \{a\} \sqsupset_F^{\text{c-Condef}} E$.

"$\tau = \mathsf{Unatt}$": Since $a \notin E \cup E_F^+$, $a \in \mathsf{Unatt}(E)$ and adding a does not remove any argument from E_F^+, $E_F^+ \subseteq (E \cup \{a\})_F^+$. Thus, $|\mathsf{Unatt}_F(E \cup \{a\})| < |\mathsf{Unatt}_F(E)|$ and $E \cup \{a\} \sqsupset_F^{\text{c-Unatt}} E$. □

Proposition 23. $\sqsupseteq^{\text{c-}\tau}$ *satisfies* addition robustness *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Unatt}\}$

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ such that $E \sqsupseteq_F^{\text{c-}\tau} E'$ with respect to $\tau \in \{\mathsf{Conflicts}, \mathsf{Unatt}\}$. Consider AF $F' = (A, R \cup \{(a, b)\}$ such that $a \in E$, $b \notin E$, and $b \in E'$.

"$\tau = \mathsf{Conflicts}$": Since $b \notin E$, $|\mathsf{Conflicts}_F(E)| = |\mathsf{Conflicts}_{F'}(E)|$, and $|\mathsf{Conflicts}_F(E')| \leq |\mathsf{Conflicts}_{F'}(E')|$, so if $E \sqsupseteq_F^{\text{c-Conflicts}} E'$ then $E \sqsupseteq_{F'}^{\text{c-Conflicts}} E'$.

"$\tau = \mathsf{Unatt}$": For every attack r in R, $r \in R \cup \{(a, b)\}$. Hence, for every argument $c \in A, c \neq a$, and $c_F^+ = c_{F'}^+$. So, $|\mathsf{Unatt}_F(E)| \leq |\mathsf{Unatt}_{F'}(E)|$ and $|\mathsf{Unatt}_F(E')| = |\mathsf{Unatt}_{F'}(E')|$. Thus, if $E \sqsupseteq_F^{\text{c-Unatt}} E'$ then $E \sqsupseteq_{F'}^{\text{c-Unatt}} E'$. □

Proposition 24. *For AF $F = (A, R)$, $A \sqsupseteq_F^\tau E$ for any $E \subset A$ with $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$.*

Proof. Let $F = (A, R)$ be any AF and $E \subset A$ a set of arguments.

"$\sqsupseteq^{\mathsf{nonatt}}$": For every argument $a \in E$, which not attacked by A it has to hold that $a_F^- = \emptyset$, so a has to be unattacked. Since $E \subset A$ and $a \in A$, therefore the number of unattacked arguments inside E is smaller or equal to the number of unattacked arguments in A. Thus, $A \sqsupseteq_F^{\mathsf{nonatt}} E$ for every $E \subset A$.

"$\sqsupseteq^{\mathsf{strdef}}$": For set E to contain any strongly defended arguments from A, E needs to contain an unattacked argument. Like already discussed in the case above, A contains every unattacked argument as well. Additionally, if an argument is strongly defended by E from A, we can use the same reasoning to show the strong defence by A from E. Therefore, the number of strongly defended arguments by E is lower or equal to the number of strongly defended arguments in A. Hence, $A \sqsupseteq_F^{\mathsf{strdef}} E$ for every $E \subset A$. □

A.4 TECHNICAL PROOFS OF CHAPTER 6

Proposition 25. r-ad *satisfies* ad-generalisation*.*

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. To show ad-*generalisation* is satisfied, we prove that if and only if $\mathsf{Conflicts}_F(E) = \mathsf{Undef}_F(E) = \emptyset$ then $E \in \mathsf{ad}(F)$.

"ad-soundness": First, $\emptyset$ is always an admissible set and therefore $\mathsf{Conflicts}_F(\emptyset) = \mathsf{Undef}_F(\emptyset) = \emptyset$, hence for every set E that is a most plausible set according to r-ad, $\mathsf{Conflicts}_F(E) = \mathsf{Undef}_F(E) = \emptyset$, otherwise $\emptyset \sqsupset_F^{\text{r-ad}} E$. Thus, for every $E \in \mathtt{max}_{\text{r-ad}}(F)$, E contains no conflict and every argument is defended, hence E is admissible.

"ad-completeness": Suppose E is admissible, then E contains no conflicts implying $\mathsf{Conflicts}_F(E) = \emptyset$ and defends all its elements $\mathsf{Undef}_F(E) = \emptyset$. There cannot be a $E' \subseteq A$ such that $E' \sqsupset_F^{\mathsf{Conflicts}} E$ or $E' \sqsupset_F^{\mathsf{Undef}} E$, because there is no subset of $\emptyset$, implying $E \in \mathtt{max}_{\text{r-ad}}(F)$. □

Proposition 26. r-ad *satisfies* respect conflicts*.*

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Then $E \sqsupset_F^{\mathsf{Conflicts}} E'$ and therefore $E \sqsupset_F^{\mathsf{r\text{-}ad}} E'$. □

Proposition 27. r-ad *satisfies* composition *and* decomposition*.*

Proof. Follows from Proposition 16 together with Definition 43. □

Proposition 28. r-ad *satisfies weak reinstatement.*

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$. Then $\mathsf{Conflicts}_F(E \cup \{a\}) = \mathsf{Conflicts}_F(E)$. What remains is to show is: $\mathsf{Undef}_F(E \cup \{a\}) \subseteq \mathsf{Undef}_F(E)$. Suppose $x \in \mathsf{Undef}_F(E \cup \{a\})$. Then $x \in E \cup \{a\}$ and $x \notin \mathfrak{F}_F(E \cup \{a\})$. Because $a \in \mathfrak{F}_F(E \cup \{a\})$ it follows that $x \in E$. Furthermore, since $x \notin \mathfrak{F}_F(E \cup \{a\})$, $x \notin \mathfrak{F}_F(E)$. This implies that $x \in \mathsf{Undef}_F(E)$. Thus, $E \cup \{a\} \equiv_F^{\mathsf{Conflicts}} E$ and $E \cup \{a\} \sqsupseteq_F^{\mathsf{Undef}} E$, therefore $E \cup \{a\} \sqsupseteq_F^{\mathsf{r\text{-}ad}} E$. □

Proposition 29. r-ad *satisfies* addition robustness*.*

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ with $E \sqsupseteq_F^{\mathsf{r\text{-}ad}} E'$. Let $F' = (A, R \cup \{(a, b)\}$ for $a \in E$, and $b \in E' \setminus E$.

"$\sqsupseteq_F^{\mathsf{Conflicts}}$": The addition of (a, b) does not add any new conflicts into E, $\mathsf{Conflicts}_F(E) = \mathsf{Conflicts}_{F'}(E)$, additionally no conflict from E' is deleted, thus $\mathsf{Conflicts}_F(E') \subseteq \mathsf{Conflicts}_{F'}(E')$. Since $E \sqsupseteq_F^{\mathsf{r\text{-}ad}} E'$, $\mathsf{Conflicts}_F(E) \subseteq \mathsf{Conflicts}_F(E')$, therefore $\mathsf{Conflicts}_{F'}(E) \subseteq \mathsf{Conflicts}_{F'}(E')$, implying $E \sqsupseteq_{F'}^{\mathsf{Conflicts}} E'$.

"$\sqsupseteq_F^{\mathsf{Undef}}$": Assume $E \sqsupseteq_F^{\mathsf{r\text{-}ad}} E'$ and $\mathsf{Conflicts}_F(E) = \mathsf{Conflicts}_F(E')$. Like show above, $\mathsf{Conflicts}_{F'}(E) \subseteq \mathsf{Conflicts}_{F'}(E')$, assume $\mathsf{Conflicts}_{F'}(E) = \mathsf{Conflicts}_{F'}(E')$. It remains to show that, $\mathsf{Undef}_{F'}(E) \subseteq \mathsf{Undef}_{F'}(E')$. Since $E \sqsupseteq_F^{\mathsf{r\text{-}ad}} E'$ and $\mathsf{Conflicts}_F(E) = \mathsf{Conflicts}_F(E')$, $\mathsf{Undef}_F(E) \subseteq \mathsf{Undef}_F(E')$. The attack (a, b) does not disable any defence from E, since $b \notin E$. Hence, b cannot be used to defend anything. Therefore, $\mathsf{Undef}_{F'}(E) \subseteq \mathsf{Undef}_F(E)$ and we do not lose any defence with respect to E.

Next, we show that E' cannot defend more argument than before. Assume $b' \in \mathfrak{F}_{F'}(E')$ and $b' \notin \mathfrak{F}_F(E')$, meaning there is one attack $(c, b') \in R$ such that E' does not attack c. Since only (a, b) is added, this attack is responsible for the defends of b' in F', therefore $a \in E'$ has to hold. However, this is a contradiction to $\mathsf{Conflicts}_{F'}(E) = \mathsf{Conflicts}_{F'}(E')$, thus b' cannot be defended. Therefore, $\mathsf{Undef}_F(E') \subseteq \mathsf{Undef}_{F'}(E')$ and since $\mathsf{Undef}_F(E) \subseteq \mathsf{Undef}_{F'}(E')$, $\mathsf{Undef}_{F'}(E) \subseteq \mathsf{Undef}_{F'}(E')$. Thus, $E \sqsupseteq_{F'}^{\mathsf{r\text{-}ad}} E'$. □

Proposition 30. r-co *satisfies* co-generalisation*.*

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$.

"co-**soundness**": Suppose $E \in \mathtt{max}_{\text{r-co}}(F)$. We first prove that E is admissible. Suppose E is not admissible, then there is an admissible E' such that $E' \sqsupset_F^{\text{r-ad}} E$. But this contradicts $E \in \mathtt{max}_{\text{r-co}}(F)$, hence, E is admissible. Next, we prove that E contains all defended arguments. Suppose the contrary, then there is an argument $a \notin E$ and a is defended by E. Thus, $a \in \mathsf{Condef}_F(E)$. For some complete extension E', $\mathsf{Condef}_F(E') = \emptyset \subset \mathsf{Condef}_F(E)$, implying $E' \equiv_F^{\text{r-ad}} E$ and $E' \sqsupset_F^{\text{r-co}} E$, contradicting $E \in \mathtt{max}_{\text{r-co}}(F)$. Hence, E contains all arguments it defends. It follows that E is complete.

"co-**completeness**": If E is a complete extension of F. Suppose, towards contradiction, that $E \notin \mathtt{max}_{\text{r-co}}(F)$. Then there is an E' such that $E' \sqsupset_F^{\text{r-co}} E$. Since E is admissible, it follows that E' is admissible. This implies $E' \sqsupset_F^{\mathsf{Condef}} E$ and hence $\mathsf{Condef}_F(E') \subset \mathsf{Condef}_F(E)$. Consequently, there is an argument $a \in \mathsf{Condef}_F(E)$, implying that there is an argument defended by E but not element of E, this contradicts the assumption that E is complete. Thus, $E \in \mathtt{max}_{\text{r-co}}(F)$. □

Proposition 31. r-co *satisfies* respect conflicts.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Then Conflicts implies, $E \sqsupset_F^{\mathsf{Conflicts}} E'$, therefore $E \sqsupset_F^{\text{r-co}} E'$. □

Proposition 32. r-co *satisfies* composition *and* decomposition.

Proof. Follows from Proposition 16 together with Definition 44. □

Proposition 33. r-co *satisfies* strong reinstatement.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Let $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$. Proposition 28 implies that $\{a\} \cup E \sqsupseteq_F^{\text{r-ad}} E$. If $\{a\} \cup E \sqsupset_F^{\text{r-ad}}, E$ we are done. Hence, we suppose $\{a\} \cup E \equiv_F^{\text{r-ad}} E$ and prove that $\mathsf{Condef}_F(\{a\} \cup E) \subset \mathsf{Condef}_F(E)$. Since $a \in \mathfrak{F}_F(E)$ and $a \notin \{E^- \cup E^+\}$, $a \in \mathfrak{F}^*(E)$ and $a \in \mathfrak{F}^*(E \cup \{a\})$. $\mathfrak{F}^*$ reaches a fixed point, implying $\mathfrak{F}^*(\{a\} \cup E) = \mathfrak{F}^*(E)$. Since $a \notin E$, $\mathfrak{F}^*(\{a\} \cup E) \setminus (E \cup \{a\}) \subset \mathfrak{F}^*(E) \setminus E$ and thus $\mathsf{Condef}_F(\{a\} \cup E) \subset \mathsf{Condef}_F(E)$. This implies $E \cup \{a\} \sqsupset_F^{\text{r-co}} E$. □

Proposition 34. r-gr *satisfies* gr-generalisation.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$.

"gr-**soundness**": Suppose $E \in \mathtt{max}_{\text{r-gr}}(F)$, then E is complete. Suppose E is not grounded. For the grounded extension E' of F, we have $E' \subset E$, which implies $E' \sqsupset_F^{\text{r-gr}} E$. But this contradicts $E \in \mathtt{max}_{\text{r-gr}}(F)$. Hence, E is the grounded extension.

"gr-**completeness**": Suppose E is the grounded extension of F and $E' \sqsupset_F^{\text{r-gr}} E$. Since E is complete, E' is also complete. Hence, $E' \subset E$, but this is impossible as it implies, E is not the grounded extension. Thus, $E \in \mathtt{max}_{\text{r-gr}}(F)$. □

Proposition 35. r-gr *satisfies* respect conflicts*,* composition*,* decomposition*, and* strong reinstatement*.* r-gr *violates* addition robustness.

Proof. ***respect conflicts*****:** Follows directly from Proposition 31.

***composition* and *decomposition*:** Follows from Proposition 16 together with Definition 45.

***strong reinstatement*:** Follows directly from Proposition 33.

***addition robustness*:** We can use Example 54, to show that r-gr violates *addition robustness*. □

Proposition 36. r-pr *and* r-co-pr *are satisfying* pr-generalisation.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. We start with r-pr.

"pr-soundness": Assume $E \in \mathtt{max}_{\text{r-pr}}(F)$. Then E is an admissible set. Suppose E is not preferred. Then there is an admissible set E' such that $E \subset E'$. But this is a contradiction to $E \in \mathtt{max}_{\text{r-pr}}(F)$. Thus, E is a preferred extension.

"pr-completeness": Suppose E is a preferred extension of F and assume $E' \sqsupset_F^{\text{r-pr}} E$. Since E is an admissible set, E' is also an admissible set. Hence, we have $E \subset E'$. But this is impossible as it implies that E is not a preferred extension. Thus, $E \in \mathtt{max}_{\text{r-pr}}(F)$.

Next, we look at r-co-pr.

"pr-soundness": Suppose $E \in \mathtt{max}_{\text{r-co-pr}}(F)$. Then E is complete. Assume E is not preferred. Then there is an complete E' such that $E \subset E'$. But this is a contradiction to $E \in \mathtt{max}_{\text{r-co-pr}}(F)$. Thus, E is a preferred extension.

"pr-completeness": Suppose E is a preferred extension of F and assume $E' \sqsupset_F^{\text{r-co-pr}} E$. Since E is complete, E' is also complete. Hence, we have $E \subset E'$. But this is impossible as it implies that E is not a preferred extension. Thus, $E \in \mathtt{max}_{\text{r-co-pr}}(F)$. □

Proposition 37. r-pr *and* r-co-pr *are satisfying* respect conflicts.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Then $E \sqsupset_F^{\text{Conflicts}} E'$, therefore $E \sqsupset_F^{\text{r-pr}} E'$ and $E \sqsupset_F^{\text{r-co-pr}} E'$ □

Proposition 38. r-pr *and* r-co-pr *are satisfying* composition *and* decomposition.

Proof. Follows from Proposition 16 together with Definition 46 and Definition 47, respectively. □

Proposition 39. r-pr *and* r-co-pr *are satisfying* strong reinstatement.

Proof. We only show it for $\mathsf{r\text{-}pr}$, but the proof for $\mathsf{r\text{-}co\text{-}pr}$ is similar.

Let $F = (A, R)$ be an AF and $E \subseteq A$. Let $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$. Proposition 28 implies, that $\{a\} \cup E \sqsupseteq_F^{\mathsf{r\text{-}pr}} E$. It also hold that $E \subset E \cup \{a\}$. Hence, $E \cup \{a\} \sqsupset_F^{\mathsf{r\text{-}pr}} E$. So, the preferred extension-ranking semantics satisfies *strong reinstatement*. □

Proposition 40. *Let $F = (A, R)$ be an AF with* $\mathsf{st}(F) \neq \emptyset$*, then* $\mathtt{max}_{\mathsf{r\text{-}sst}}(F) = \mathsf{st}(F)$.

Proof. Let $F = (A, R)$ be an AF such that $\mathsf{st}(F) \neq \emptyset$.

"$\subseteq$": Let $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$ and $E' \in \mathsf{st}(F)$, meaning that $\mathsf{Conflicts}_F(E) = \mathsf{Undef}_F(E) = \mathsf{Condef}_F(E) = \emptyset$ and $E' \in \mathsf{co}(F)$, implying $\mathsf{Conflicts}_F(E') = \mathsf{Undef}_F(E') = \mathsf{Condef}_F(E') = \emptyset$. Additionally, E' attacks every argument not inside, hence $\mathsf{Unatt}_F(E') = \emptyset$, so for $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$ it has to hold that $\mathsf{Unatt}_F(E) = \emptyset$, meaning that E is conflict-free and attacks everything not inside, implying that E is a stable extension.

"$\supseteq$": Let $E \in \mathsf{st}(F)$, this means that E is conflict-free and attacks every argument not in E, i.e., $\mathsf{Conflicts}_F(E) = \mathsf{Unatt}_F(E) = \emptyset$. Additionally, E is complete, therefore $\mathsf{Undef}_F(E) = \mathsf{Condef}_F(E) = \emptyset$. There cannot be any set be ranked strictly more plausible to be accepted than E with respect to $\mathsf{r\text{-}sst}$, thus $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$. □

Proposition 41. $\mathsf{r\text{-}sst}$ *satisfies* st-completeness

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. If $\mathsf{st}(F) = \emptyset$, then $\mathtt{max}_{\mathsf{r\text{-}sst}}(F) \supseteq \emptyset$.

Suppose $E \in \mathsf{st}(F)$, then $\mathsf{Conflicts}_F(E) = \emptyset$ and $\mathsf{Unatt}_F(E) = \emptyset$, implying that $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$, thus st-*completeness* is satisfied. □

Proposition 42. $\mathsf{r\text{-}sst}$ *satisfies* sst-generalisation.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$.

"sst-soundness": Suppose $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$. Then E is complete. Assume E is not semi-stable. Then there is a complete set E' such that $E \cup E^+ \subset E' \cup E'^+$. But, this implies $E' \sqsupset_F^{\mathsf{Unatt}} E$, contradicting $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$. Hence, E is a semi-stable extension.

"sst-completeness": Suppose E is a semi-stable extension of F and assume $E' \sqsupset_F^{\mathsf{r\text{-}sst}} E$. Since E is complete, E' is also complete. Hence, $E' \sqsupset_F^{\mathsf{Unatt}} E$. But this is impossible as it implies that $E \cup E^+ \subset E' \cup E'^+$, meaning that E is not a semi-stable extension. Thus, $E \in \mathtt{max}_{\mathsf{r\text{-}sst}}(F)$. □

Proposition 43. $\mathsf{r\text{-}sst}$ *satisfies* respect conflicts*,* composition*,* decomposition*, and* strong reinstatement. $\mathsf{r\text{-}sst}$ *violates* addition robustness.

Proof. ***respect conflicts*:** Follows directly from Proposition 31.

***composition* and *decomposition*:** Follows from Proposition 16 together with Definition 48.

***strong reinstatement*:** Follows directly from Proposition 33.

***addition robustness*:** We can use Example 54, to show that r-sst violates *addition robustness*.

□

Proposition 44. *Let $F = (A, R)$ be an AF and $E, E' \subseteq A$, if $E \sqsupseteq_F^{\text{r-}\sigma} E'$ then $E \sqsupseteq_F^{\text{r-c-}\sigma} E'$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$.*

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$, assume $E \sqsupseteq_F^{\text{r-}\sigma} E'$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{sst}\}$, then there is one $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ such that $\tau_F(E) \subseteq \tau_F(E')$, this subset behaviour entails $|\tau_F(E)| \leq |\tau_F(E')|$, thus $E \sqsupseteq_F^{\text{r-c-}\sigma} E'$.

For $\sigma \in \{\mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}\}$, we still use $\subseteq$ and $\supseteq$, respectively, so there is no difference between these variants. □

Proposition 45. r-c-σ *satisfies* σ-generalisation $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$.

Proof. For any $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, the best value a set can have is $\emptyset$. For the cardinality-based versions of these relations, the best value to be returned is 0, and only the empty set can have cardinality of 0. Therefore, we know that if and only if τ returns $\emptyset$, the cardinality-based versions of r-c-σ returns 0. Thus, we can use the same proof ideas of r-σ to show that r-c-σ satisfies σ-*generalisation*. □

Proposition 46. r-c-σ *satisfies* respect conflicts *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Then $E \sqsupset_F^{\text{c-Conflicts}} E'$ and $E \sqsupset_F^{\text{r-c-}\sigma} E'$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$. □

Proposition 47. r-c-σ *satisfies* composition *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$.

Proof. Let $F_1 = (A_1, R_1)$, $F_2 = (A_2, R_2)$ and $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ be AFs with $A_1 \cap A_2 = \emptyset$. We know that $|\tau_{F_1}(E \cap A_1)| + |\tau_{F_2}(E \cap A_2)| = |\tau_F(E)|$ for any base relation $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$. Assume $E \cap A_1 \sqsupseteq_{F_1}^{\text{r-c-}\sigma} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\text{r-c-}\sigma} E' \cap A_2$. Thus, there is at least one $\tau' \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$ such that $|\tau'_{F_1}(E \cap A_1)| \leq |\tau'_{F_1}(E' \cap A_1)|$ and $|\tau'_{F_2}(E \cap A_2)| \leq |\tau'_{F_2}(E' \cap A_2)|$. Therefore, $|\tau'_{F_1}(E \cap A_1)| + |\tau'_{F_2}(E \cap A_2)| \leq |\tau'_{F_1}(E' \cap A_1)| + |\tau'_{F_2}(E' \cap A_2)|$ and this entails $|\tau'_F(E)| \leq |\tau'_F(E')|$. Hence, *composition* is satisfied. □

Proposition 48. r-c-ad *satisfies* weak reinstatement *and violates* strong reinstatement.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin E^- \cup E^+$. In Proposition 17, we have shown that $\mathsf{Undef}_F(E \cup \{a\}) \subseteq \mathsf{Undef}_F(E)$ and therefore also $|\mathsf{Undef}_F(E \cup \{a\})| \leq |\mathsf{Undef}_F(E)|$. This proves that r-c-ad satisfies *weak reinstatement*.

Since ad-*generalisation* is satisfied, we know that *strong reinstatement* is violated.

□

Proposition 49. $\mathsf{r\text{-}c\text{-}}\sigma$ *satisfies* strong reinstatement *for* $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin E^- \cup E^+$. In Proposition 17, we have shown that $\mathsf{Condef}_F(E \cup \{a\}) \subset \mathsf{Condef}_F(E)$. This also shows that $|\mathsf{Condef}_F(E \cup \{a\})| < |\mathsf{Condef}_F(E)|$. Hence, $\mathsf{r\text{-}c\text{-}}\sigma$ satisfies *strong reinstatement* and because $E \cup \{a\} \sqsupset_F^{\mathsf{r\text{-}c\text{-}}\sigma} E$, this also shows the satisfaction of *strong reinstatement* for the remaining semantics $\sigma \in \{\mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$. $\square$

Proposition 50. $\mathsf{r\text{-}c\text{-}ad}$ *satisfies* addition robustness.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ with $E \sqsupseteq_F^{\mathsf{r\text{-}c\text{-}ad}} E'$. Let $F' = (A, R \cup \{(a, b)\})$ for $a \in E$ and $b \notin E$, $b \in E'$. The addition of (a, b) does not add any new conflict into E and does not remove any conflict from E', therefore $|\mathsf{Conflicts}_F(E)| = |\mathsf{Conflicts}_{F'}(E)|$ and $|\mathsf{Conflicts}_F(E')| \leq |\mathsf{Conflicts}_{F'}(E')|$. Hence, *addition robustness* is satisfied by $\sqsupseteq^{\mathsf{c\text{-}Conflicts}}$.

It remains to show, that *addition robustness* holds for $\sqsupseteq^{\mathsf{c\text{-}Undef}}$ as well. We already know that no defence in E is removed, therefore $|\mathsf{Undef}_{F'}(E)| \leq |\mathsf{Undef}_F(E)|$ and E' cannot defend more arguments in F' than in F, otherwise additional conflicts have to be added. Thus, $E \sqsupseteq_{F'}^{\mathsf{r\text{-}c\text{-}ad}} E'$. $\square$

Proposition 51. *Let* $F = (A, R)$ *be an AF.* $\sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)}$ *is* reflexive *and* transitive *for any sequence of base relations* $(\tau_1, \ldots, \tau_n)$.

Proof. Let $F = (A, R)$ be an AF and $(\tau_1, \ldots, \tau_n)$ a sequence of base relations.

"reflexive": For any $E \subseteq A$ it has to hold that $E \equiv_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E$. Let w_i be the number of sets ranked worse than E with respect to $\sqsupseteq_F^{\tau_i}$ and l_i be the number of sets ranked better than E with respect to $\sqsupseteq_F^{\tau_i}$, then for $E \equiv_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E$ to hold, $w_i - l_i = w_i - l_i$ has to hold for every $1 \leq i \leq n$, which is clear by definition.

"transitive": Let $E, E', E'' \subseteq A$ and $E \sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E'$ and $E' \sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E''$, then E has a better win/loss record than E' and E' has a better win/loss record than E''. Let w, w', w'' be the corresponding number of wins minus the number of loses of E, E' and E'', respectively. Thus, $w > w'$ and $w' > w''$, by definition $w > w''$, which shows that $E \sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E''$. $\square$

Proposition 52. *Let* $F = (A, R)$ *be an AF. If base relation* $\sqsupseteq^\tau$ *is* total *and* transitive, *then* $\mathsf{cope}(\sqsupseteq_F^\tau) = \sqsupseteq_F^\tau$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. Let $\sqsupseteq^\tau$ be a base relation that is *total* and *transitive*. First, we show that if $E \sqsupseteq_F^\tau E'$ then $E \sqsupseteq_F^{\mathsf{cope}(\tau)} E'$. Assume $E \sqsupseteq_F^\tau E'$, then for every E'' such that $E' \sqsupseteq_F^\tau E''$, because of transitivity, we have $E \sqsupseteq_F^\tau E''$. So, E wins at least as many times as E'. If $E'' \sqsupseteq_F^\tau E$, then $E'' \sqsupseteq_F^\tau E'$, so E' loses at least as often as E. Thus, $|\{\mathcal{E} \subseteq A | E \sqsupseteq_F^\tau \mathcal{E}\}| - |\{\mathcal{E}' \subseteq A | \mathcal{E}' \sqsupseteq_F^\tau E\}| \geq |\{\mathcal{E} \subseteq A | E' \sqsupseteq_F^\tau \mathcal{E}\}| - |\{\mathcal{E}' \subseteq A | \mathcal{E}' \sqsupseteq_F^\tau E'\}|$, implying $E \sqsupseteq_F^{\mathsf{cope}(\tau)} E'$.

Next, we show the other direction. Let $E \sqsupseteq_F^{\mathsf{cope}(\tau)} E'$, and assume for contrary $E' \sqsupset_F^{\tau} E$, similarly to what we showed above, this implies $E' \sqsupseteq_F^{\mathsf{cope}(\tau)} E$ and because of transitivity and totality of τ, the relationship has to be strict, $E' \sqsupset_F^{\mathsf{cope}(\tau)} E$. Thus, $E \sqsupseteq_F^{\mathsf{cope}(\tau)} E'$ can only imply $E \sqsupseteq_F^{\tau} E'$ or $E \asymp_F^{\tau} E'$. However, the second option is not possible since $\sqsupseteq_F^{\tau}$ is total. So, $E \sqsupseteq_F^{\mathsf{cope}(\tau)} E'$ implies $E \sqsupseteq_F^{\tau} E'$. $\square$

Proposition 53. $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ *satisfies* ad-generalisation.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$.

"ad-***completeness***": Suppose E is admissible, then $\mathsf{Conflicts}_F(E) = \mathsf{Undef}_F(E) = \emptyset$ and therefore $E \sqsupseteq_F^{\mathsf{Conflicts}} E'$ and $E \sqsupseteq_F^{\mathsf{Undef}} E'$ for any $E' \subseteq A$. So, E wins against every set of arguments and E only loses against sets $\mathcal{E}$ with $\mathsf{Conflicts}_F(\mathcal{E}) = \emptyset$ and $\mathsf{Undef}_F(\mathcal{E}) = \emptyset$, respectively. However, all these sets are also losing against E, there is no set with less loses than E. E wins the maximal amount of times and no set loses less than E, therefore there cannot be any set with a better win/ loss record than E, implying $E \in \mathtt{max}_{\mathsf{cope}(\mathsf{Conflicts},\mathsf{Undef})}(F)$.

"ad-***soundness***": Suppose $E \in \mathtt{max}_{\mathsf{cope}(\mathsf{Conflicts},\mathsf{Undef})}(F)$, so $E \sqsupseteq_F^{\mathsf{cope}(\mathsf{Conflicts},\mathsf{Undef})} \emptyset$. We know that $\mathsf{Conflicts}(\emptyset) = \mathsf{Undef}(\emptyset) = \emptyset$, meaning that $\emptyset$ wins the maximal amount of times and there is no set which loses less than $\emptyset$, i. e., $\emptyset \in \mathtt{max}_{\mathsf{cope}(\mathsf{Conflicts},\mathsf{Undef})}(F)$. For $E \in \mathtt{max}_{\mathsf{cope}(\mathsf{Conflicts},\mathsf{Undef})}(F)$ to hold E has to have the same win/ loss record as $\emptyset$, which is only possible if $\mathsf{Conflicts}(E) = \mathsf{Undef}(E) = \emptyset$, implying that E is admissible. $\square$

Proposition 54. $\mathsf{cope}(\sqsupseteq^{\mathsf{Conflicts}}, \sqsupseteq^{\mathsf{Undef}})$ *satisfies* weak reinstatement.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Consider $a \in \mathcal{F}_F, a \notin E$, and $a \notin (E^- \cup E^+)$. First, we examine $\sqsupseteq_F^{\mathsf{Conflicts}}$ and $\sqsupseteq_F^{\mathsf{Undef}}$ one by one.

"$\sqsupseteq^{\mathsf{Conflicts}}$": Since $a \notin (E^- \cup E^+)$, $\mathsf{Conflicts}_F(E) = \mathsf{Conflicts}_F(E \cup \{a\})$. Hence, if $E \sqsupseteq_F^{\mathsf{Conflicts}} E'$ for any $E' \subseteq A$ then $E \cup \{a\} \sqsupseteq_F^{\mathsf{Conflicts}} E'$ holds as well. Thus, the win/ loss record of E and $E \cup \{a\}$ are the same.

"$\sqsupseteq^{\mathsf{Undef}}$": Since $a \in \mathcal{F}_F(E)$, $\mathsf{Undef}(E) = \mathsf{Undef}(E \cup \{a\})$. Thus, again the win/ loss records of these two sets are equal.

Next, let $w_{\mathsf{Conflicts}}$ and w_{Undef} be the win/ loss records of E and $w'_{\mathsf{Conflicts}}$ and w'_{Undef} the win/ loss records of $E \cup \{a\}$. Since $w_{\mathsf{Conflicts}} = w'_{\mathsf{Conflicts}}$ and $w_{\mathsf{Undef}} = w'_{\mathsf{Undef}}$, it holds that $w_{\mathsf{Conflicts}} + w_{\mathsf{Undef}} = w'_{\mathsf{Conflicts}} + w'_{\mathsf{Undef}}$. Thus, $E \cup \{a\} \equiv_F^{\mathsf{cope}(\mathsf{Conflicts},\mathsf{Undef})} E$ and *weak reinstatement* is satisfied. $\square$

Proposition 55. $\mathsf{cope}(\sqsupseteq^{\tau})$ *satisfies* strong reinstatement *for* $\tau \in \{\mathsf{nonatt}, \mathsf{strdef}\}$.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Consider $a \in \mathfrak{F}_F(E), a \notin E$, and $a \notin (E^- \cup E^+)$.

"$\tau = \mathsf{nonatt}$": We show that if $E \sqsupseteq_F^{\mathsf{nonatt}} E'$ then $E \cup \{a\} \sqsupseteq_F^{\mathsf{nonatt}} E'$ for $E' \subseteq A$. Every argument that is attacked by E is also attacked by $E \cup \{a\}$, i.e., $E^+ \subseteq (E \cup \{a\})^+$. If a is attacked by E', since $a \in \mathfrak{F}_F(E)$, the attacker of a is attacked by E, hence the addition of a does not add any new attacked arguments, meaning $E \cup \{a\} \sqsupseteq_F^{\mathsf{nonatt}} E'$ if $E \sqsupseteq_F^{\mathsf{nonatt}} E'$, implying that $E \cup \{a\}$ wins at least as often as E.

Next, we show that E loses at least as often as $E \cup \{a\}$. Assume $E' \sqsupseteq_F^{\mathsf{nonatt}} E \cup \{a\}$, then like discussed before every argument attacked by E' in $E \cup \{a\}$ is also attacked by E' in E except a, however since a is defended by E this argument does not create any problem, so $E' \sqsupseteq_F^{\mathsf{nonatt}} E$. Thus, the win/lose ratio of $E \cup \{a\}$ is at least as good as E and also $E \cup \{a\}$ wins against E so $E \cup \{a\} \sqsupset_F^{\mathsf{cope(nonatt)}} E$.

"$\tau = \mathsf{strdef}$": Similarly to the $\sqsupseteq^{\mathsf{nonatt}}$ case, every argument attacked by E is also attacked by $E \cup \{a\}$, hence the same reasoning can be utilised. Thus, the addition of a cannot lower the number of arguments strongly defended by $E \cup \{a\}$ from E', so $E \cup \{a\} \sqsupseteq_F^{\mathsf{cope(strdef)}} E'$ if $E \sqsupseteq_F^{\mathsf{cope(strdef)}} E'$. Therefore the win/lose record of $E \cup \{a\}$ is at least as good as the win/lose record of E and since $a \notin E_F^-$, a is not attacked by E and therefore strongly defended by $E \cup \{a\}$ from E, implying $E \cup \{a\} \sqsupset_F^{\mathsf{strdef}} E$. Thus, $E \cup \{a\} \sqsupset_F^{\mathsf{cope(strdef)}} E$. □

A.5 TECHNICAL PROOFS OF CHAPTER 7

Proposition 56. *If κ_F^Z is defined, then κ_F^Z satisfies AF F.*

Proof. Let $F = (A, R)$ be an AF. In order to show that κ_F^Z satisfies F, we need to show, that κ_F^Z satisfies both principles of an OCF, i.e. acceptance of attacks and possibly reinstating every argument.

accept attack. Let $a, b \in A$ such that $(a, b) \in R$ with $a \neq b$ is an attack, it has to hold that $\kappa_F^Z(\{a\}) < \kappa_F^Z(\{a, b\})$. We know that $\kappa_F^Z(\{a\}) = 0$, because $\{a\}$ can only violate the attack (a, a), which cannot exist. Hence, it is enough to show, that $\kappa_F^Z(\{a, b\}) > 0$. Since, (a, b) exists, $\{a, b\}$ violates this attack, and thus $\kappa_F^Z(\{a, b\}) > 0$.

argument possibly reinstated. Let $a \in A$ be an argument. Assume $\kappa_F^Z(S \cup a) > \kappa^Z(S)$ for some $S \subseteq A$ with $S \cap (a_F^- \cup a_F^+) = \emptyset$. This is only possible, if $S \cup \{a\}$ violates an attack $r \in R$ and S does not violate r. There is one argument $b \in S$ such that $r = (a, b)$ or $r = (b, a)$, and $a \neq b$. Hence, $b \in a_F^- \cup a_F^+$. Because of $S \cap (a_F^- \cup a_F^+) = \emptyset$, there cannot be such an argument $b \in S$ with $b \in a_F^- \cup a_F^+$. Thus, $\kappa_F^Z(S \cup a) > \kappa_F^Z(S)$ is impossible. □

Proposition 57. *κ_F^Z is undefined for AF F with $\mathsf{st}(F) = \emptyset$.*

Proof. Let $F = (A, R)$ be an AF. We show the contraposition, if κ^Z_F is defined for F, then $\mathsf{st}(F) \neq \emptyset$. Let κ^Z_F be defined. We can find a Z-attack-Partitioning $(R_0, ..., R_n)$. For every attack r in R_0, there is a set E such that r is verified, every attack is satisfied and attack admissibility of every argument $a \in A$ is satisfied. Next, we show that E is stable. E has to be conflict-free, otherwise there is an attack, which is violated. We show that every argument not in E is attacked by E, thus $E \cup E^+_F = A$. Let $b \notin E$, because attack admissibility of b is satisfied, there is an argument $c \in b^-_F$ with $c \in E$, hence there is an attacker of b which is part of E. □

Proposition 58. *For $F = (A, R)$ and Z-attack-Partitioning $(R_0, \ldots, R_n)$ of R, if $(a, b) \in R_0$ then $a \in \mathsf{cred}_{\mathsf{ad}}(F)$.*

Proof. Let $F = (A, R)$ be an AF, $(R_0, ..., R_n)$ a Z-attack-Partitioning of R and $(a, b) \in R_0$, then (a, b) is tolerated by R, meaning that there is an $E \subseteq A$ such that (a, b) is verified, each attack in R is satisfied by E, and attack admissibility of each argument $c \in A$ is satisfied by E. To verify (a, b), it holds that $a \in E$ and $b \notin E$. Also, it has to hold for all $c \in a^-_F$ that $c \notin E$. All attackers of a are *out*. Next, we show that E is admissible. E is conflict-free, otherwise, one attack would be violated. For every $d \in E$, no attacker of d is in E. To not violate attack admissibility, for every $e \notin E$ at least one attacker of e has to be in E, meaning that E attacks every argument not contained in E. Hence, for every attacker b of an argument $a \in E$, we have an argument $d \in E$ such that d attacks b. E is admissible, and therefore a is part of some admissible extension of F, thus $a \in \mathsf{cred}_{\mathsf{ad}}(F)$. □

Proposition 59. $\sqsupseteq^{\kappa^Z}$ *satisfies* respect conflicts *and* σ-completeness *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof. Consider AF $F = (A, R)$ and $E, E' \subseteq A$ such that $E \in \mathsf{cf}(F)$ and $E' \notin \mathsf{cf}(F)$. Since E is conflict-free, E cannot violate any attack and therefore $\kappa^Z_F(E) = 1$, implying $E \in \mathtt{max}_{\kappa^Z}(F)$. For E', there has to be at least one attack $r \in R$ that is violated by E', so $\kappa^Z_F(E') > 1$, this implies $E \sqsupset^{\kappa^Z}_F E'$.

Consider $E \in \sigma(F)$, then $E \in \mathsf{cf}(F)$ and therefore $E \in \mathtt{max}_{\kappa^Z}(F)$ implying that *σ-completeness* for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ is satisfied. □

Proposition 60. $\sqsupseteq^{\kappa^Z}$ *satisfies* composition.

Proof. Let $F = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and $E, E' \subseteq A_1 \cup A_2$. For *composition*, we need to show that, if $\kappa^Z_{F_1}(E \cap A_1) \leq \kappa^Z_{F_1}(E' \cap A_1)$ and $\kappa^Z_{F_2}(E \cap A_2) \leq \kappa^Z_{F_2}(E' \cap A_2)$ then $\kappa^Z_F(E) \leq \kappa^Z_F(E')$. By definition of κ^Z, $\kappa^Z_F(E)$ is the maximal value between $\kappa^Z_{F_1}(E \cap A_1)$ and $\kappa^Z_{F_2}(E \cap A_2)$, because if an attack r_1 is violated by E, then r_1 is also violated by either $E \cap A_1$ or $E \cap A_2$. Similar holds for $\kappa^Z(E')$. We have to check four possible cases for $max(\kappa^Z_{F_1}(E \cap A_1), \kappa^Z_{F_2}(E \cap A_2)) \leq max(\kappa^Z_{F_1}(E' \cap A_1), \kappa^Z_{F_2}(E' \cap A_2))$.

1. $\kappa^Z_{F_1}(E \cap A_1) \leq \kappa^Z_{F_1}(E' \cap A_1)$

2. $\kappa^Z_{F_2}(E \cap A_2) \leq \kappa^Z_{F_2}(E' \cap A_2)$
3. $\kappa^Z_{F_1}(E \cap A_1) \leq \kappa^Z_{F_2}(E' \cap A_2)$
4. $\kappa^Z_{F_2}(E \cap A_2) \leq \kappa^Z_{F_1}(E' \cap A_1)$

Case 1 and 2 are clear by definition. For case 3, $\kappa^Z_{F_1}(E \cap A_1) \geq \kappa^Z_{F_2}(E \cap A_2)$ and $\kappa^Z_{F_1}(E' \cap A_1) \leq \kappa^Z_{F_2}(E' \cap A_2)$, but also $\kappa^Z_{F_1}(E \cap A_1) \leq \kappa^Z_{F_1}(E' \cap A_1)$, which proves case 3. Case 4 is similar to case 3. $\square$

Proposition 61. $\sqsupseteq^{\kappa^Z}$ *satisfies* weak reinstatement.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Assume $a \notin E$ and $a \notin (E^-_F \cup E^+_F)$. We have to show that $\kappa^Z_F(E \cup \{a\}) \leq \kappa^Z_F(E)$. $E \cup \{a\}$ violates the same attacks as E, since E and $\{a\}$ are not in a conflict with each other. This means, that $\kappa^Z_F(E \cup \{a\})$ cannot be greater than $\kappa^Z_F(E)$. This implies $E \cup \{a\} \sqsupseteq^{\kappa^Z}_F E$. $\square$

Proposition 62. $\sqsupseteq^{\kappa^Z}$ *satisfies* addition robustness.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$ with $E \sqsupseteq^{\kappa^Z}_F E'$. Suppose $F' = (A, R \cup \{(a, b)\}$ for $a \in E$, and $b \in E' \setminus E$.

Since $\{a, b\} \not\subseteq E$, E verifies attack (a, b) and we do not add any more attacks into F', thus we get: $\kappa^Z_F(E) = \kappa^Z_{F'}(E)$. We do not remove any attack from F', so if E' violates an attack in F then E' violates the same attack in F', thus $\kappa^Z_F(E) \leq \kappa^Z_{F'}(E)$. This implies $E \sqsupseteq^{\kappa^Z}_{F'} E'$. $\square$

A.6 TECHNICAL PROOFS OF CHAPTER 8

Proposition 63. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics. $\sqsupseteq^{\mathsf{Hoare}_\rho}_F$ is reflexive and transitive.*

Proof. Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics.

"reflexive": Let $E \subseteq A$, since argument-ranking semantics ρ induce a preorder, for every $a \in E$, it holds that $a \simeq^\rho_F a$. Therefore, for every argument a there is an argument a ranked at least as good. Implying, $E \equiv^{\mathsf{Hoare}_\rho}_F E$.

"transitive": $E, E', E'' \subseteq A$, and ρ be an argument-ranking semantics. Assume $E \sqsupseteq^{\mathsf{Hoare}_\rho}_F E'$ and $E' \sqsupseteq^{\mathsf{Hoare}_\rho}_F E''$. Then for every argument $a \in E''$ there is an argument $b \in E'$ such that $b \succeq^\rho_F a$. Since $E \sqsupseteq^{\mathsf{Hoare}_\rho}_F E'$, for this argument b, there is an argument $c \in E$ such that $c \succeq^\rho_F b$ and therefore also $c \succeq^\rho_F a$. Thus, it holds that $E \sqsupseteq^{\mathsf{Hoare}_\rho}_F E''$. $\square$

Proposition 64. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics, then $A \in \mathtt{max}_{\mathsf{Hoare}_\rho}(F)$.*

Proof. Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics. W.l.o.g. let $a \in A$ be the best ranked argument with respect to ρ, i. e., $a \succeq_F^\rho b$ for any $b \in A$. Then for every set of arguments $E \subset A$ argument a is ranked at least as good as every argument of E and $a \in A$, therefore $A \sqsupseteq_F^{\mathsf{Hoare}_\rho} E$ and $A \in \mathtt{max}_{\mathsf{Hoare}_\rho}(F)$ □

Proposition 65. *If ρ satisfies* In*, then* Hoare_ρ *satisfies* composition*.*

Proof. Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ and argument-ranking semantics ρ. Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{Hoare}_\rho} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{Hoare}_\rho} E' \cap A_2$. Then for all $b \in E' \cap A_1$, there is an argument $a \in E \cap A_1$ such that $a \succeq_{F_1}^\rho b$. If ρ satisfies *In*, then $a \succeq_F^\rho b$ as well. Similar holds for every $b' \in E' \cap A_2$. For every argument $b \in E'$, there is an argument $a \in E$ such that $a \succeq_F^\rho b$, thus $E \sqsupseteq_F^{\mathsf{Hoare}_\rho} E'$. □

Proposition 66. *If ρ satisfies* VP*,* NaE*, and* In*, then* Hoare_ρ *violates* decomposition*.*

Proof. Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ be an AF with $A_1 \cap A_2 = \emptyset$ and ρ an argument-ranking semantics satisfying *VP*, *NaE*, and *In*. Suppose $a, c \in A_1$ and $b, d \in A_2$, such that a and b are unattacked, i. e., $a_F^- = b_F^- = \emptyset$, and c, d are attacked. Then *VP* and *NaE* imply:

$$a \simeq_F^\rho b \succ_F^\rho c \quad \text{and} \quad a \simeq_F^\rho b \succ_F^\rho d$$

Therefore:

$$\{a, d\} \equiv_F^{\mathsf{Hoare}_\rho} \{b, c\}$$

In implies:

$$a \succ_{F_1}^\rho c \quad \text{and} \quad b \succ_{F_2}^\rho d$$

Therefore:

$$\{a\} \sqsupset_{F_1}^{\mathsf{Hoare}_\rho} \{c\} \quad \text{and} \quad \{b\} \sqsupset_{F_2}^{\mathsf{Hoare}_\rho} \{d\}$$

This shows that *decomposition* is violated if *VP*, *NaE*, and *In* are satisfied. □

Proposition 67. Hoare_ρ *satisfies* weak reinstatement*.*

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E), a \notin E$, and $a \notin (E^- \cup E^+)$. Then there are two cases. First, assume $a \succ_F^\rho b$ for every $b \in E$, then $E \cup \{a\} \sqsupset_F^{\mathsf{Hoare}_\rho} E$. Next, assume there is an argument $b \in E$ such that $b \succ_F^\rho a$. Since ρ is reflexive, for every argument $c \in A$, $c \succeq_F^\rho c$ and therefore $E \cup \{a\} \equiv_F^{\mathsf{Hoare}_\rho} E$. □

Proposition 68. *If ρ satisfies* Abs*, then* Hoare_ρ *satisfies* syntax independence*.*

Proof. Let $F = (A, R)$ and $F' = (A', R')$ be AFs such that there is a isomorphism $\gamma : A \to A'$ with $F' = \gamma(F)$. If ρ satisfies *Abs*, for all $a, b \in A$ such that $a \succeq_F^\rho b$ then $\gamma(a) \succeq_{\gamma(F)}^\rho \gamma(b)$. The underling argument rankings are the same for F and F', implying that for two sets $E, E' \subseteq A$ with $E \sqsupseteq_F^{\mathsf{Hoare}_\rho} E'$ then $\gamma(E) \sqsupseteq_{\gamma(F)}^{\mathsf{Hoare}_\rho} \gamma(E')$. □

Proposition 69. $\sqsupseteq^{\mathsf{GCB}_{\rho,\Box}}$ *is a* total *preoder for* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.

Proof. Let $F = (A, R)$ be and ρ a gradual semantics.

"reflexive" : Assume $E \subseteq A$.

$\Box = \mathsf{sum}$: $\sum_{a \in E} \rho_F(a) = \sum_{a' \in E} \rho_F(a')$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{sum}}} E$.

$\Box = \mathsf{max}$: $\mathsf{max}(\rho_F(E)) = \mathsf{max}(\rho_F(E))$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{max}}} E$.

$\Box = \mathsf{min}$: $\mathsf{min}(\rho_F(E)) = \mathsf{min}(\rho_F(E))$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{min}}} E$.

$\Box \in \{\mathsf{leximax}, \mathsf{leximin}\}$: For every i, we have $v_i = v_i$ for $v_i \in E$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\Box}} E$.

"transitive" : Assume $E, E', E'' \subseteq A$ s.t. $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\Box}} E'$ and $E' \sqsupseteq_F^{\mathsf{GCB}_{\rho,\Box}} E''$.

$\Box = \mathsf{sum}$: $\sum_{a \in E} \rho_F(a) \geq \sum_{b \in E'} \rho_F(b) \geq \sum_{c \in E''} \rho_F(c)$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{sum}}} E''$.

$\Box = \mathsf{max}$: $\mathsf{max}(\rho_F(E)) \geq \mathsf{max}(\rho_F(E')) \geq \mathsf{max}(\rho_F(E''))$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{max}}} E''$.

$\Box = \mathsf{min}$: $\mathsf{min}(\rho_F(E)) \geq \mathsf{min}(\rho_F(E')) \geq \mathsf{min}(\rho_F(E''))$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{min}}} E''$.

$\Box = \mathsf{leximax}$: Let $\rho_F(E) = \langle \rho_F(a_1), \ldots, \rho_F(a_n) \rangle$, $\rho_F(E') = \langle \rho_F(a'_1), \ldots, \rho_F(a'_m) \rangle$, and $\rho_F(E'') = \langle \rho_F(a''_1), \ldots, \rho_F(a''_n) \rangle$ be in descending order. Then there is an argument $a_i \in E$ such that $\rho_F(a_i) > \rho_F(a'_i)$ and for all $j < i$: $\rho_F(a_j) = \rho_F(a'_j)$ and there is an $a'_k \in E'$ such that $\rho_F(a'_k) > \rho_F(a''_k)$ and for all $l < k$: $\rho_F(a'_l) = \rho_F(a''_l)$. If $i \leq k$, then $\rho_F(a_i) > \rho_F(a'_i) = \rho_F(a''_i)$ and if $i > k$ then $\rho_F(a_k) = \rho_F(a'_k) > \rho_F(a''_k)$. Thus, $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximax}}} E''$.

$\Box = \mathsf{leximin}$: Let $\rho_F(E) = \langle \rho_F(a_1), \ldots, \rho_F(a_n) \rangle$, $\rho_F(E') = \langle \rho_F(a'_1), \ldots, \rho_F(a'_m) \rangle$, and $\rho_F(E'') = \langle \rho_F(a''_1), \ldots, \rho_F(a''_n) \rangle$ be in ascending order. Then there is an argument $a_i \in E$ such that $\rho_F(a_i) > \rho_F(a'_i)$ and for all $j < i$: $\rho_F(a_j) = \rho_F(a'_j)$ and there is an $a'_k \in E'$ such that $\rho_F(a'_k) > \rho_F(a''_k)$ and for all $l < k$: $\rho_F(a'_l) = \rho_F(a''_l)$. If $i \leq k$, then $\rho_F(a_i) > \rho_F(a'_i) = \rho_F(a''_i)$ and if $i > k$ then $\rho_F(a_k) = \rho_F(a'_k) > \rho_F(a''_k)$. Thus, $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximin}}} E''$.

"total" : Assume $E, E' \subseteq A$.

$\Box = \mathsf{sum}$: $\sum_{a \in E} \rho_F(a)$ and $\sum_{b \in E'} \rho_F(b)$ are both real numbers, thus get have $\sum_{a \in E} \rho_F(a) \geq \sum_{a \in E'} \rho_F(b)$ or $\sum_{a \in E} \rho_F(a) \leq \sum_{a \in E'} \rho_F(b)$.

$\Box = \mathsf{max}$: $\mathsf{max}(\rho_F(E)$ and $\mathsf{max}(\rho_F(E')$ are both real numbers, thus get have $\mathsf{max}(\rho_F(E) \geq \mathsf{max}(\rho_F(E')$ or $\mathsf{max}(\rho_F(E) \leq \mathsf{max}(\rho_F(E')$.

$\Box = \mathsf{min}$: $\mathsf{min}(\rho_F(E)$ and $\mathsf{min}(\rho_F(E')$ are both real numbers, thus get have $\mathsf{min}(\rho_F(E) \geq \mathsf{min}(\rho_F(E')$ or $\mathsf{min}(\rho_F(E) \leq \mathsf{min}(\rho_F(E')$.

$\Box = \mathsf{leximax}$: For every pair of argument $a_i \in E$ and $b_i \in E'$, we can decide if $\rho_F(a_i) \leq \rho_F(b_i)$ or $\rho_F(a_i) \geq \rho_F(b_i)$. Thus, either $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximax}}} E'$ or $E' \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximax}}} E$.

$\square =$ leximin: For every pair of argument $a_i \in E$ and $b_i \in E'$, we can decide if $\rho_F(a_i) \leq \rho_F(b_i)$ or $\rho_F(a_i) \geq \rho_F(b_i)$. Thus, either $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximin}}} E'$ or $E' \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximin}}} E$. $\square$

Proposition 70. *For an AF $F = (A, R)$, sets $E, E' \subseteq A$, and a gradual semantics ρ, it holds that for $\square \in \{\mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$:*

$$E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\square}} E' \text{ if and only if } E \sqsupseteq_F^{\mathsf{GCB}_{sv^{\rho},\square}} E'$$

Proof. Let AF $F = (A, R)$, $E, E' \subseteq A$, ρ be a gradual semantics, and $\square \in \{\mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. Suppose $E = \{e_1, e_2, \ldots, e_n\}$ and $E' = \{e'_1, e'_2, \ldots, e'_m\}$. Their corresponding vectors are:

$$\rho(E) = \langle \rho(e_1), \rho(e_2), \ldots, \rho(e_n) \rangle \quad \text{and} \quad \rho(E') = \langle \rho(e'_1), \rho(e'_2), \ldots, \rho(e'_m) \rangle$$

Assume $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\square}} E'$, if $\rho(e_i) \geq \rho(e'_j)$ then $e_i \succeq_F^{\rho} e'_j$ and therefore $sv_F^{\rho}(e_i) \geq sv_F^{\rho}(e'_j)$. Thus, the relationships between each pair of arguments remains the same. For $\square = \mathsf{max}$:

$$\mathsf{max}(\langle \rho(e_1), \rho(e_2), \ldots, \rho(e_n) \rangle) = \mathsf{max}(\langle sv_F^{\rho}(e_1), sv_F^{\rho}(e_2), \ldots, sv_F^{\rho}(e_n) \rangle)$$

The same reasoning holds for $\square \in \{\mathsf{min}, \mathsf{leximin}, \mathsf{leximax}\}$.

Assume $E \sqsupseteq_F^{\mathsf{GCB}_{sv^{\rho},\square}} E'$, if $sv_F^{\rho(e_i)} \geq sv_F^{\rho(e'_j)}$ then $e_i \succeq_F^{\rho} e'_j$, so it has to hold that $\rho(e_i) \geq \rho(e'_j)$. Thus, the relationships between each pair of arguments remains the same. Implying $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\square}} E'$. $\square$

Proposition 71. *Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics, then $\sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{max}}} = \sqsupseteq_F^{\mathsf{Hoare}_{\rho}}$.*

Proof. Let $F = (A, R)$ be an AF, $E, E' \subseteq A$, and ρ an argument-ranking semantics. Note that because of Proposition 70, it is enough to show the statement for argument-ranking semantics is general and do not need to restrict to gradual semantics.

Assume $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{max}}} E'$, then $\mathsf{max}(\rho_F(E)) \geq \mathsf{max}(\rho(E'))$, so for every argument $b \in E'$ there is at least one argument $a \in E$ such that $\rho_F(a) \geq \rho_F(b)$, in particular $\rho_F(a) \geq \mathsf{max}(\rho_F(E'))$ and because of transitivity of ρ, we have $\mathsf{max}(\rho_F(E)) \geq \rho(b)$. Therefore $E \sqsupseteq_F^{\mathsf{Hoare}_{\rho}} E'$.

Assume $E \sqsupseteq_F^{\mathsf{Hoare}_{\rho}} E'$, then for all $b \in E'$ there is an argument $a \in E$ such that $\rho_F(a) \geq \rho_F(b)$. Especially, there is an argument $a' \in E$ with $\rho_F(a') \geq \mathsf{max}(\rho_F(E'))$ and because of the transitivity of ρ_F, we have $\mathsf{max}(\rho_F(E)) \geq \rho_F(a') \geq \mathsf{max}(\rho_F(E'))$. Therefore $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{max}}} E'$. $\square$

Proposition 72. *Let $F = (A, R)$ be an AF, then $A \in \mathtt{max}_{\mathsf{GCB}_{\rho,\square}}(F)$ for $\square \in \{\mathsf{sum}, \mathsf{leximax}\}$ and argument-ranking semantics ρ.*

Proof. Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics.

"$\square$ = sum": Since the corresponding strength values of every argument $a \in A$ is in $[0,1]$, we know that $\rho(a) \geq 0$ and therefore $\sum_{a\in A} \rho_F(A) \geq \sum_{b\in E} \rho_F(b)$ for every $E \subseteq A$. Hence, $A \in \mathtt{max}_{\mathsf{GCB}_{\rho,\square}}(F)$.

"$\square$ = leximax": Let $\rho_F(A) = \langle \rho_F(a_1), \ldots,$ $\rho_F(a_n)\rangle$ be in descending order, then for any $E \subset A$ there is at least one a_i missing from E. Let $\rho_F(E) = \langle \rho_F(a_1), \ldots, \rho_F(a_{i-1}), \rho_F(a_{i+1,}), \ldots, \rho_F(a_n)\rangle$, then $\rho_F(a_i) \geq \rho_F(a_{i+1})$, which are compared after i steps. In the $i+1$ step $\rho_F(a_{i+1})$ and $\rho_F(a_{i+2})$ are compared. At some point either the previous element is strictly bigger or $\rho_F(a_n)$ is compared to the blank element. Therefore, $A \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximax}}} E$ and thus $A \in \mathtt{max}_{\mathsf{GCB}_{\rho,\mathsf{leximax}}}(F)$. $\square$

Lemma 4. *Let V, W, X, Y be vectors sorted in descending order such that $V \sqsupseteq^{\mathsf{leximax}} W$ and $X \sqsupseteq^{\mathsf{leximax}} Y$. It holds that $c_1 * V \cup c_2 * X \sqsupseteq^{\mathsf{leximax}} c_1 * W \cup c_2 * Y$ for $c_1, c_2 \in \mathbb{N}$.*

Proof. Let $V = \langle v_1, v_2, \ldots\rangle, W = \langle w_1, w_2, \ldots\rangle, X = \langle x_1, x_2, \ldots\rangle, Y = \langle y_1, y_2, \ldots\rangle$ be vectors in descending order s. t. $V \sqsupseteq^{\mathsf{leximax}} W$ and $X \sqsupseteq^{\mathsf{leximin}} Y$.

First, we multiply the constants c_1 and c_2 with the individual vectors, i. e., $c_1 * V = \langle c_1 * v_1, c_1 * v_2, \ldots\rangle, c_2 * W = \langle c_2 * w_1, c_2 * w_2, \ldots\rangle, c_1 * X = \langle c_1 * x_1, c_1 * x_2, \ldots\rangle, c_2 * Y = \langle c_2 * y_1, c_2 * y_2, \ldots\rangle$.

We normalise the length of these vectors by appending $-\infty$ to shorter vectors, i. e., for two vectors $A = \langle a_1, \ldots, a_{m_1}\rangle, B = \langle b_1, \ldots, b_{m_2}\rangle$ s. t. $|A| > |B|$ we add $-\infty$ until both vectors have the same length $B' = \langle b_1, b_2, \ldots, b_{m_2}, -\infty, \ldots\rangle$ such that $|A| = |B'|$. This addition does not change the order between A and B, which we show in the following. Assume $A \sqsupseteq^{\mathsf{leximax}} B$, since $|A| > |B|$ these two vectors cannot be equal. So, $A \sqsupset^{\mathsf{leximax}} B$, therefore there exists an i such that $a_i = b_i$ for $i = 1, \ldots, m$ and $a_{i+1} > b_{i+1}$. Since B and B' are equal up to position m_2, these pairs are also in B', hence $A \sqsupset^{\mathsf{leximax}} B'$. Next, assume $B \sqsupset^{\mathsf{leximax}} A$ then there exists an i such that $a_i = b_i$ for $i = 1, \ldots m$ and $b_{i+1} > a_{i+1}$. For $b_j \in B$, $b_j \in B'$, implying $B' \sqsupset^{\mathsf{leximax}} A$. We can assume w.l.o.g. $|c_1 * V| = |c_1 * W| = |c_2 * X| = |c_2 * Y|$. Let $c_1 * V \sqsupseteq^{\mathsf{leximax}} c_1 * W$, $c_2 * X \sqsupseteq^{\mathsf{leximax}} c_2 * Y$ and $c_1 * V \cup c_2 * X = \langle u_1, u_2, \ldots\rangle$ and $c_1 * W \cup c_2 * Y = \langle z_1, z_2, \ldots\rangle$.

We reformulate the definition of lexicographic order: Let $A = \langle a_1, a_2, \ldots\rangle, B = \langle b_1, b_2, \ldots\rangle$ be vectors in descending order $A \sqsupseteq^{\mathsf{leximax}} B \Leftrightarrow (a_1 > b_1) \vee (A^1_\downarrow \sqsupseteq^{\mathsf{leximax}} B^1_\downarrow \wedge a_1 = b_1)$, where $A^i_\downarrow$ is the vector where the first i elements of A are removed, $A^i_\downarrow = \langle a_{i+1}, \ldots\rangle$.

We show $c_1 * V \cup c_2 * X \sqsupseteq^{\mathsf{leximax}} c_1 * W \cup c_2 * Y$ in four cases.

Case: $c_1 * V \sqsupset^{\mathsf{leximax}} c_1 * W$ and $c_2 * X \sqsupset^{\mathsf{leximax}} c_2 * Y$: There exists an i such that $i = 1, \ldots m_1$, $c_1 * v_i = c_1 * w_i$, and $c_1 * v_{i+1} > c_1 * w_{i+1}$ and there exists an j such that $j = 1, \ldots m_2$, $c_2 * x_j = c_2 * y_j$, and $c_2 * x_{j+1} > c_2 * y_{j+1}$, where $m_1, m_2 \in \mathbb{N}$. We modify the vectors such that $m_1 = m_2$, so the difference between the vectors occurs at the same step. W.l.o.g. assume $m_1 > m_2$, then we add $m_1 - m_2$ times ∞ at the beginning of X and Y, such that the first differences in these vectors occur at m_1. This modification does not change the order, as we argued above. After this modification, we have to make sure, that $|V| = |W| = |X| = |Y|$ still holds.

The next step will be done via induction for $m_1 \to |V|$. Case $m_1 = 1$: We have: $c_1 * v_1 > c_1 * w_1$ and $c_2 * x_1 > c_2 * y_1$. If $u_1 = c_1 * v_1$ and $z_1 = c_1 * w_1$ or $u_1 = c_2 * x_1$ and $z_1 = c_2 * y_1$, then it is clear that $u_1 > z_1$. If $u_1 = c_1 * v_1$ and $z_1 = c_2 * y_1$ or $u_1 = c_2 * x_1$ and $z_1 = c_1 * w_1$, then $c_1 * v_1 \geq c_2 * x_1 > c_2 * y_1$ and $c_2 * x_1 \geq c_1 * v_1 > c_1 * w_1$, respectively. This shows that $u_1 > z_1$ and therefore $V \cup X \sqsupset^{\mathsf{leximax}} W \cup Y$.

Assume that for all $m_1 = 1, \ldots, n$ it holds that $u_{m_1} = z_{m_1}$.

Case $n \to n+1$: Since for all $i = 1, \ldots, n$ we have $u_i = z_i$, it is enough to look at the $i+1$ elements, so $\langle c_1 * V \cup c_2 * X \rangle^i_\downarrow = \langle u_{i+1}, \ldots \rangle$ and $\langle c_1 * W \cup c_2 * Y \rangle^i_\downarrow = \langle z_{i+1}, \ldots \rangle$. If $u_{i+1} \in c_1 * V$ and $z_{i+1} \in c_1 * W$ or $u_{i+1} \in c_2 * X$ and $z_{i+1} \in c_2 * Y$, then $u_{i+1} > z_{i+1}$ since $c_1 * v_{i+1} > c_1 * w_{i+1}$ and $c_2 * x_{i+1} > c_2 * y_{i+1}$, respectively. If $u_{i+1} \in c_1 * V$ and $z_{i+1} \in c_2 * Y$ or $u_{i+1} \in c_2 * X$ and $z_{i+1} \in c_1 * W$, then we have $c_1 * v_{i+1} \geq c_2 * x_{i+1} > c_2 * y_{i+1}$ and $c_2 * x_{i+1} \geq c_1 * v_{i+1} > c_1 * w_{i+1}$, respectively. This implies $c_1 * V \cup c_2 * X \sqsupset^{\mathsf{leximax}} c_1 * W \cup c_2 * Y$.

Case: $c_1 * V \equiv^{\mathsf{leximax}} c_1 * W$ **and** $c_2 * X \sqsupset^{\mathsf{leximax}} c_2 * Y$**:** There exists an i such that $i = 1, \ldots m_1$ $c_2 * x_i = c_2 * y_i$ and $c_2 * x_{i+1} > c_2 * y_{i+1}$ and for every $j = 1, \ldots, |c_1 * V|$, $c_1 * v_j = c_1 * w_j$. We show $c_1 * V \cup c_2 * X \sqsupset^{\mathsf{leximax}} c_1 * W \cup c_2 * Y$ via induction for $m \to |V|$.

Case $m_1 = 1$: Then we have $c_2 * x_1 > c_2 * y_1$. If $u_1 = c_1 * v_1$ and $z_1 = c_1 * w_1$, then $u_1 = z_1$. If $u_1 = c_2 * x_1$ and $z_1 = c_2 * y_1$, then $u_1 > z_1$. If $u_1 = c_1 * v_1$ and $z_1 = c_2 * y_1$, then $c_1 * v_1 \geq c_2 * x_1 > c_2 * y_1$. If $u_1 = c_2 * x_1$ and $z_1 = c_1 * w_1$, then $c_2 * x_1 \geq c_1 * v_1 = c_1 * w_1$. So, $u_1 \geq z_1$ in all cases.

Assume that for all $m_1 = 1, \ldots, n$ it holds that $u_{m_1} = z_{m_1}$.

Case $n \to n+1$: Since for all $i = 1, \ldots, n$, $u_i = z_i$, it is enough to examine the $i+1$ elements, so $\langle c_1 * V \cup c_2 * X \rangle^i_\downarrow = \langle u_{i+1}, \ldots \rangle$ and $\langle c_1 * W \cup c_2 * Y \rangle^i_\downarrow = \langle z_{i+1}, \ldots \rangle$. If $u_{i+1} = c_1 * v_{i+1}$ and $z_{i+1} = c_1 * w_{i+1}$, then $u_{i+1} = z_{i+1}$. If $u_{i+1} = c_2 * x_{i+1}$ and $z_{i+1} = c_2 * y_{i+1}$, then $u_{i+1} > z_{i+1}$, since $c_2 * xi + 1 > c_2 * y_{i+1}$. If $u_{i+1} = c_1 * v_{i+1}$ and $z_{i+1} = c_2 * y_{i+1}$, then $c_1 * v_{i+1} \geq c_2 * x_{i+1} > c_2 * y_{i+1}$. If $u_{i+1} = c_2 * x_{i+1}$ and $z_{i+1} = c_1 * w_{i+1}$, then $c_2 * x_{i+1} \geq c_1 * v_{i+1} = c_1 * w_{i+1}$. This implies $c_1 * V \cup c_2 * X \sqsupset^{\mathsf{leximax}} c_2 * W \cup c_2 * Y$.

Case: $c_1 * V \sqsupset^{\mathsf{leximax}} c_1 * W$ **and** $c_2 * X \equiv^{\mathsf{leximax}} c_2 * Y$**:** We can use the same reasoning as for case $c_1 * V \equiv^{\mathsf{leximax}} c_1 * W$ and $c_2 * X \sqsupset^{\mathsf{leximax}} c_2 * Y$.

Case: $c_1 * V \equiv^{\mathsf{leximax}} c_1 * W$ **and** $c_2 * X \equiv^{\mathsf{leximax}} c_2 * Y$**:** For all $i = 1, \ldots, |c_1 * V|$, we have $c_1 * v_i = c_1 * w_i$ and $c_1 * x_i = c_1 * y_i$. We show $c_1 * V \cup c_2 * X \sqsupseteq^{\mathsf{leximax}} c_1 * W \cup c_2 * Y$ via induction for $m_1 \to |c_1 * V|$. Case $m_1 = 1$: If $u_1 = c_1 * v_1$ and $z_1 = c_1 * w_1$ or $u_1 = c_2 * x_1$ and $z_1 = c_2 * y_1$, then $u_1 = z_1$. If $u_1 = c_1 * v_1$

and $z_1 = c_2 * y_1$ or $u_1 = c_2 * x_1$ and $z_1 = c_1 * w_1$, then $c_1 * v_1 \geq c_2 * x_1 = c_2 * y_1$ and $c_1 * x_1 \geq c_1 * v_1 = c_1 * w_1$, , respectively.

Assume that for all $m_1 = 1, \ldots, n$ it holds that $u_{m_1} = z_{m_1}$. Case $n \to n+1$: Since for all $i = 1, \ldots, n$, we have $u_i = z_i$, it is enough to examine the $i+1$ elements, so $\langle c_1 * V \cup c_2 * X \rangle^i_{\downarrow} = \langle u_{i+1}, \ldots \rangle$ and $\langle c_1 * W \cup c_2 * Y \rangle^i_{\downarrow} = \langle z_{i+1}, \ldots \rangle$.

If $u_{i+1} = c_1 * v_{i+1}$ and $z_{i+1} = c_1 * w_{i+1}$ or $u_{i+1} = c_2 * x_{i+1}$ and $z_{i+1} = c_2 * y_{i+1}$, then $u_{i+1} = z_{i+1}$. If $u_{i+1} = c_1 * v_{i+1}$ and $z_{i+1} = c_2 * y_{i+1}$ or $u_{i+1} = c_2 * x_{i+1}$ and $z_{i+1} = c_1 * w_{i+1}$, then $c_1 * v_{i+1} \geq c_2 * x_{i+1} = c_2 * y_{i+1}$ and $c_2 * x_{i+1} \geq c_1 * v_{i+1} = c_1 * w_{i+1}$, respectively. This implies $c_1 * V \cup c_2 * X \sqsupseteq^{\mathsf{leximax}} c_1 * W \cup c_2 * Y$. □

Lemma 5. *Let V, W, X, Y be vectors sorted in ascending order such that $V \sqsupseteq^{\mathsf{leximin}} W$ and $X \sqsupseteq^{\mathsf{leximin}} Y$. It holds that $c_1 * V \cup c_2 * X \sqsupseteq^{\mathsf{leximin}} c_1 * W \cup c_2 * Y$ for $c_1, c_2 \in \mathbb{N}$.*

Proof. We can use the same reasoning as for Lemma 4. □

Proposition 73. *If ρ satisfies* Independence for gradual semantics*, then* $\mathsf{GCB}_{\rho,\Box}$ *satisfies* composition *for* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$.

Proof. Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and ρ a gradual semantics that satisfies *Independence for gradual semantics*.

"$\Box = \mathsf{sum}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{sum}}}_{F_1} E' \cap A_1$ and $E \cap A_2 \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{sum}}}_{F_2} E' \cap A_2$. If ρ satisfies *Independence for gradual semantics*, then $\rho_{F_1}(a) = \rho_F(a)$ for all $a \in A_1$ same holds for all $b \in A_2$. We get:

$$\sum_{a \in E \cap A_1} \rho_{F_1}(a) \geq \sum_{b \in E' \cap A_1} \rho_{F_1}(b) \quad \text{and} \quad \sum_{a \in E \cap A_2} \rho_{F_2}(a) \geq \sum_{b \in E' \cap A_2} \rho_{F_2}(b)$$

Then we have:

$$\sum_{a \in E \cap A_1} \rho_{F_1}(a) + \sum_{a' \in E \cap A_2} \rho_{F_2}(a') \geq \sum_{b \in E' \cap A_1} \rho_{F_1}(b) + \sum_{b' \in E' \cap A_2} \rho_{F_2}(b')$$

This implies that, $\sum_{a \in E} \rho_F(a) \geq \sum_{b \in E'} \rho_F(b)$, thus $E \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{sum}}}_{F} E'$.

"$\Box = \mathsf{leximax}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{leximax}}}_{F_1} E' \cap A_1$ and $E \cap A_2 \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{leximax}}}_{F_2} E' \cap A_2$. If ρ satisfies *Independence for gradual semantics*, then $\rho_{F_1}(a) = \rho_F(a)$ for all $a \in A_1$ same holds for all $b \in A_2$. We utilise Lemma 4 with $c_1 = c_2 = 1$ to show, that $E \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{leximax}}}_{F} E'$.

"$\Box = \mathsf{min}$": $E, E' \subseteq A_1 \cup A_2$ such that $E \cap A_1 \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{min}}}_{F_1} E' \cap A_1$ and $E \cap A_2 \sqsupseteq^{\mathsf{GCB}_{\rho,\mathsf{min}}}_{F_2} E' \cap A_2$ and let ρ be an gradual semantics that satisfies *Independence for gradual semantics*. Then $\mathsf{min}(\rho_{F_1}(E \cap A_1)) \geq \mathsf{min}(\rho_{F_1}(E' \cap A_1))$ and $\mathsf{min}(\rho_{F_2}(E \cap A_2)) \geq \mathsf{min}(\rho_{F_2}(E' \cap A_2))$. W.l.o.g. let $\mathsf{min}(\rho_{F_1}(E \cap A_1)) \geq \mathsf{min}(\rho_{F_2}(E \cap A_2))$, so $\mathsf{min}(\rho_F(E)) = \mathsf{min}(\rho_{F_1}(E \cap A_1))$. If $\mathsf{min}(\rho_{F_1}(E' \cap A_1)) \geq \mathsf{min}(\rho_{F_2}(E' \cap A_2))$, we are done since $\mathsf{min}(\rho_{F_1}(E \cap A_1)) \geq \mathsf{min}(\rho_{F_1}(E' \cap A_1))$. $\mathsf{min}(\rho_{F_1}(E' \cap A_1)) \leq \mathsf{min}(\rho_{F_2}(E' \cap A_2))$, then $\mathsf{min}(\rho_{F_2}(E \cap A_2)) \geq \mathsf{min}(\rho_{F_2}(E' \cap A_2)) \geq \mathsf{min}(\rho_{F_1}(E' \cap A_1))$, thus $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{min}}} E'$.

"$\square = \mathsf{leximin}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{\rho,\mathsf{leximin}}} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_1}^{\mathsf{GCB}_{\rho,\mathsf{leximin}}} E' \cap A_2$. If ρ satisfies *Independence for gradual semantics*, then $\rho_{F_1}(a) = \rho_F(a)$ for all $a \in A_1$ same holds for all $b \in A_2$. So, we utilise Lemma 5 with $c_1 = c_2 = 1$ to show that $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{leximin}}} E'$. □

Proposition 74. *If ρ satisfies* Independence for gradual semantics*,* Equivalence*, and* Maximality*, then* $\mathsf{GCB}_{\rho,\square}$ *violates* decomposition *for* $\square \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximax}\}$.

Proof. Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ be an AF and ρ a gradual semantics that satisfies *Independence for gradual semantics*, *Equivalence*, and *Maximality*. Let $a, c \in A_1$ and $b, d \in A_2$ with a and b are unattacked, i. e. $a_F^- = b_F^- = \emptyset$, and c, d are attacked. Then $\rho_F(a) = \rho_F(b) > \rho_F(c)$ and $\rho_F(a) = \rho_F(b) > \rho_F(d)$. Consider the two sets $\{a, d\}$ and $\{b, c\}$, these two sets are comparable, so:

$$\{a, d\} \sqsupseteq_F^{\mathsf{GCB}_{\rho,\square}} \{b, c\} \quad \text{or} \quad \{b, c\} \sqsupseteq_F^{\mathsf{GCB}_{\rho,\square}} \{a, d\}$$

for $\square \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximax}\}$. However, $\rho_F(a) = \rho_{F_1}(a) > \rho_F(c) = \rho_{F_1}(c)$ and $\rho_F(b) = \rho_{F_2}(b) > \rho_F(d) = \rho_{F_2}(d)$, thus

$$\{a\} \sqsupset_{F_1}^{\mathsf{GCB}_{\rho,\square}} \{c\} \quad \text{and} \quad \{b\} \sqsupset_{F_2}^{\mathsf{GCB}_{\rho,\square}} \{d\}$$

for all $\square \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximax}\}$. Thus, if ρ satisfies *Independence for gradual semantics*, *Equivalence*, and *Maximality*, then $\mathsf{GCB}_{\rho,\square}$ violates *decomposition* for $\square \in \{\mathsf{sum}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximax}\}$. □

Proposition 75. $\mathsf{GCB}_{\rho,\square}$ *satisfies* weak reinstatement *for* $\square = \mathsf{sum}$ *and* strong reinstatement *for* $\square = \mathsf{leximax}$.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E), a \notin E$, and $a \notin (E^- \cup E^+)$.

"$\square = \mathsf{sum}$": For gradual semantics ρ, we have $\rho(b) \geq 0$ for every argument $b \in A$, hence $\sum_{b \in E \cup \{a\}} \rho(b) \geq \sum_{c \in E} \rho(c)$. Therefore, $E \cup \{a\} \sqsupseteq_F^{\mathsf{GCB}_{\rho,\mathsf{sum}}} E$. If $\rho(a) = 0$, then $E \cup \{a\} \equiv_F^{\mathsf{GCB}_{\rho,\mathsf{sum}}} E$ and therefore *strong reinstatement* is violated.

"$\square = \mathsf{leximax}$": Let $E \cup \{a\} = \{a_1, \ldots, a_{i-1}, a, a_i, \ldots, a_n\}$ and $E = \{a_1, \ldots, a_{i-1}, a_i, \ldots, a_n\}$. If there is a_j such that $\rho(a_j) > \rho(a_{j-1})$ for $j \geq i$, then $E \cup \{a\} \sqsupset_F^{\mathsf{GCB}_{\rho,\mathsf{leximax}}} E$. If there is no such a_j, because $|E \cup \{a\}| > |E|$, we have $E \cup \{a\} \sqsupset_F^{\mathsf{GCB}_{\rho,\mathsf{leximax}}} E$. □

Proposition 76. *If gradual semantics ρ satisfies* Abs*, then* $\mathsf{GCB}_{\rho,\square}$ *satisfies* syntax independence *for* $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$.

Proof. Let $F = (A, R)$, $F' = (A', R')$ be AFs, $E, E' \subseteq A$, and ρ a gradual semantics. Assume isomorphism $\gamma : A \to A'$. If ρ satisfies *Abs*, then $\rho(a) = \rho(\gamma(a))$ for every $a \in A$ and therefore *syntax independence* is satisfied. $\square$

Proposition 77. *Let $F = (A, R)$ be an AF, then $A \in \mathtt{max}_{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}}(F)$ for argument-ranking-semantics ρ.*

Proof. Let $F = (A, R)$ be an AF and ρ an argument-ranking semantics. For every $a \in A$, $sv^\rho_F(a) \geq 0$ and therefore $\sum_{a\in A} sv^\rho_F(a) \geq \sum_{b\in E} sv^\rho_F(b)$ for every $E \subset A$. Thus, $A \in \mathtt{max}_{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}}(F)$ $\square$

Proposition 78. $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ *satisfies* weak reinstatement *and violates* strong reinstatement.

Proof. Let $F = (A, R)$ be an AF, ρ an argument-ranking semantics, and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E), a \notin E$, and $a \notin (E^- \cup E^+)$. For every ρ, $sv^\rho_F(b) \geq 0$ for every argument $b \in A$, hence $\sum_{b\in E\cup\{a\}} sv^\rho_F(b) \geq \sum_{c\in E} sv^\rho_F(c)$. Implying, $E \cup \{a\} \sqsupseteq_F^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}} E$. If $sv^\rho_F(a) = 0$, then $E \cup \{a\} \equiv^{\mathsf{GCB}_{sv^\rho,\mathsf{sum}}} FE$. Thus, *weak reinstatement* is satisfied, while *strong reinstatement* is violated. $\square$

Proposition 79. *If ρ satisfies* Abs*, then* $\mathsf{GCB}_{sv^\rho,\mathsf{sum}}$ *satisfies* syntax independence.

Proof. Let $F = (A, R)$, $F' = (A', R')$ be AFs, $E, E' \subseteq A$, and ρ an argument-ranking semantics. Assume isomorphism $\gamma : A \to A'$. If ρ satisfies *Abs*, then $sv^\rho_F(a) = sv^\rho_{F'}(\gamma(a))$ for every $a \in A$ and therefore *syntax independence* is satisfied. $\square$

Proposition 80. *Let $F = (A, R)$ be an AF, then $A \in \mathtt{max}_{\mathsf{GCB}_{ne_\sigma,\square}}(F)$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$ and $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}\}$.*

Proof. Let $F = (A, R)$ be an AF and $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$.

"$\square = \mathsf{sum}$": Since $ne_{\sigma,F}(a) \geq 0$, for every $a \in A$, $\sum_{a\in A} ne_{\sigma,F}(a) \geq \sum_{b\in E} ne_{\sigma,F}(b)$ for any $E \subset A$, therefore $A \in \mathtt{max}_{\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}}(F)$.

"$\square = \mathsf{max}$": Let a be $\max(ne_{\sigma,F}(A))$, then there cannot be an argument $b \in A$ with $ne_{\sigma,F}(b) > ne_{\sigma,F}(a)$. For every $E \subseteq A$, $\max(ne_{\sigma,F}(E)) \leq ne_{\sigma,F}(a)$ and therefore $A \in \mathtt{max}_{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}}(F)$.

"$\square = \mathsf{leximax}$": Let $\rho_F(A) = \{\rho_F(a_1), \ldots, \rho_F(a_n)\}$ be in descending order then for any $E \subset A$ there is at least one a_i missing. Let $ne_{\sigma,F}(E) = \langle ne_{\sigma,F}(a_1), \ldots, ne_{\sigma,F}(a_{i-1}), ne_{\sigma,F}(a_{i+1}), \ldots, ne_{\sigma,F}(a_n)\rangle$, then $ne_{\sigma,F}(a_i) \geq ne_{\sigma,F}(a_{i+1})$, which are compared after i steps. In the $i + 1$ step $ne_{\sigma,F}(a_{i+1})$ and $ne_{\sigma,F}(a_{i+2})$ are compared. At some point either the previous element is strictly bigger or $ne_{\sigma,F}(a_n)$ is compared to the empty set. Therefore $A \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}} E$ and $A \in \mathtt{max}_{\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}}(F)$. $\square$

Lemma 6. *Let* $F_1 = (A_1, R_1)$ *and* $F_2 = (A_2, R_2)$ *be two AFs such that* $A_1 \cap A_2 = \emptyset$ *and* $F = F_1 \cup F_2$ *then it holds that* $|\{E \in \sigma(F) \mid a \in E\}| = |\{E \in \sigma(F_1) \mid a \in E\}| * |\sigma(F_2)| + |\{E \in \sigma(F_2) \mid a \in E\}| * |\sigma(F_1)|$ *for every* $a \in A_1 \cup A_2$ *and* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof. Let $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ be two AFs such that $A_1 \cap A_2 = \emptyset$ and $F = F_1 \cup F_2$ and $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$. W.l.o.g. assume there is a $E \in \sigma(F_1)$ with $a \in E$. Consider any set E' such that $E' \in \sigma(F_2)$ and $a \notin E'$. Every argument defended by E in F_1 is still defended by E in F, for E' the same holds for their defended arguments in F_2. So, $E, E' \in \mathsf{ad}(F)$ and additionally there cannot be any attacks between E and E'. We have $E \cup E' \in \mathsf{ad}(F)$. Hence we can combine E with every admissible set of F_2 to get a new admissible set in F. Hence, $|\{E \in \mathsf{ad}(F) \mid a \in E\}| = |\{E \in \mathsf{ad}(F_1) \mid a \in E\}| * |\mathsf{ad}(F_2)| + |\{E \in \mathsf{ad}(F_2) \mid a \in E\}| * |\mathsf{ad}(F_1)|$. Note that $|\{E \in \mathsf{ad}(F_2) \mid a \in E\}| = 0$.

The only arguments that are defended by E in F that were originally in F_2 are unattacked arguments. However these unattacked arguments have to be part of every complete extension of F_2, thus if $E \in \mathsf{co}(F_1)$ and $E' \in \mathsf{co}(F_2)$ then $E \cup E' \in \mathsf{co}(F)$. This implies $|\{E \in \mathsf{co}(F) \mid a \in E\}| = |\{E \in \mathsf{co}(F_1) \mid a \in E\}| * |\mathsf{co}(F_2)| + |\{E \in \mathsf{co}(F_2) \mid a \in E\}| * |\mathsf{co}(F_1)|$.

For $E \in \mathsf{gr}(F_1)$ and $E' \in \mathsf{gr}(F_2)$, every argument defended by E in F_1 is also defended by E in F, hence if $a \in \mathfrak{F}_{F_1}(E)$ then $a \in \mathfrak{F}_F(E)$. This implies $E \cup E' \in \mathsf{gr}(F)$ and therefore $|\{E \in \mathsf{gr}(F) \mid a \in E\}| = |\{E \in \mathsf{gr}(F_1) \mid a \in E\}| * |\mathsf{gr}(F_2)| + |\{E \in \mathsf{gr}(F_2) \mid a \in E\}| * |\mathsf{gr}(F_1)|$.

If $E \in \mathsf{pr}(F_1)$ and $E' \in \mathsf{pr}(F_2)$, then we cannot add any argument into $E \cup E'$ without violation the admissibility of this set in F. Thus, $E \cup E'$ is a preferred extensions in F. Hence $|\{E \in \mathsf{pr} \mid a \in E\}| = |\{E \in \mathsf{pr}(F_1) \mid a \in E\}| * |\mathsf{pr}(F_2)| + |\{E \in \mathsf{pr}(F_2) \mid a \in E\}| * |\mathsf{pr}(F_1)|$.

Any argument attacked by E in F_1 is also attacked by E in F and every not attacked argument by E in F_1 is also not attacked by E in F. This implies $|\{E \in \sigma'(F) \mid a \in E\}| = |\{E \in \sigma'(F_1) \mid a \in E\}| * |\sigma'(F_2)| + |\{E \in \sigma'(F_2) \mid a \in E\}| * |\sigma'(F_1)|$. for $\sigma' \in \{\mathsf{st}, \mathsf{sst}\}$. □

Proposition 81. $\mathsf{GCB}_{ne_\sigma,\Box}$ *satisfies* composition *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ *and* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{leximax}, \mathsf{min}, \mathsf{leximin}\}$.

Proof. Let $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ be two AFs such that $A_1 \cap A_2 = \emptyset$, $F = F_1 \cup F_2$ and $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

"$\Box = \mathsf{sum}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}} E' \cap A_2$, then $\sum_{a \in E \cap A_1} ne_{\sigma,F_1}(a) \geq \sum_{b \in E' \cap A_1} ne_{\sigma,F_1}(b)$ and

$\sum_{a'\in E\cap A_2} ne_{\sigma,F_2}(a') \geq \sum_{b'\in E'\cap A_2} ne_{\sigma,F_2}(b')$. Because of Lemma 6, we get:

$$\begin{aligned}
\sum_{a\in E} ne_{\sigma,F}(a) &= \sum_{a\in E} \frac{|\{S\in\sigma(F)\mid a\in S\}|}{|\sigma(F)|} \\
&= \sum_{a\in E} \frac{|\{S\in\sigma(F_1)\mid a\in S\}| * |\sigma(F_2)| + |\{S\in\sigma(F_2)\mid a\in S\}| * |\sigma(F_1)|}{|\sigma(F)|} \\
&= \frac{1}{|\sigma(F)|} * \sum_{a\in E} |\{S\in\sigma(F_1)\mid a\in S\}| * |\sigma(F_2)| \\
&\qquad + |\{S\in\sigma(F_2)\mid a\in S\}| * |\sigma(F_1)| \\
&\geq \frac{1}{|\sigma(F)|} * \sum_{b\in E'} |\{S'\in\sigma(F_1)\mid b\in S'\}| * |\sigma(F_2)| \\
&\qquad + |\{S'\in\sigma(F_2)\mid b\in S'\}| * |\sigma(F_1)| \\
&= \sum_{b\in E'} \frac{|\{S'\in\sigma(F)\mid b\in S'\}|}{|\sigma(F)|} \\
&= \sum_{b\in E'} ne_{\sigma,F}(b)
\end{aligned}$$

So, we get $E \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}} E'$.

"$\square = \mathsf{max}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E\cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}} E'\cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}} E'\cap A_2$.

Let $\mathsf{max}(ne_{\sigma,F}(E)) = \frac{|\{S\in\sigma(F_1)|a\in S\}|*|\sigma(F_2)|+|\{S\in\sigma(F_2)|a\in S\}|*|\sigma(F_1)|}{|\sigma(F)|}$ and $\mathsf{max}(ne_{\sigma,F}(E')) = \frac{|\{S'\in\sigma(F_1)|b\in S'\}|*|\sigma(F_2)|+|\{S'\in\sigma(F_2)|b\in S'\}|*|\sigma(F_1)|}{|\sigma(F)|}$.

If $a \in A_1$ and $b \in A_1$, then we compare $|\{S\in\sigma(F_1)\mid a\in S\}| * |\sigma(F_2)|$ and $|\{S'\in\sigma(F_1)\mid b\in S'\}| * |\sigma(F_2)|$. $E\cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}} E'\cap A_1$ implies that $\mathsf{max}(ne_{\sigma,F_1}(E)) = \frac{|\{S\in\sigma(F_1)|a\in S\}|}{|\sigma(F_1)|} \geq \frac{|\{S'\in\sigma(F_1)|b\in S'\}|}{|\sigma(F_1)|} = \mathsf{max}(ne_{\sigma,F_1}(E'))$

If $a \in A_2$ and $b \in A_2$, we can use the same reasoning as above.

For $a \in A_1$ and $b \in A_2$, $\frac{|\sigma(F_2)|*|\{S\in\sigma(F_1)|a\in S\}|}{|\sigma(F_1)|} \geq \frac{|\sigma(F_1)|*|\{S\in\sigma(F_2)|a'\in S\}|}{|\sigma(F_2)|}$ for every $a' \in E\cap A_2$. Let $ne_{\sigma,F_2}(a') = \mathsf{max}(ne_{\sigma,F_2}(E\cap A_2))$, then $\frac{|\sigma(F_1)|}{|\sigma(F)|} * ne_{\sigma,F_2}(a') \geq \frac{|\sigma(F_1)|}{|\sigma(F)|} * ne_{\sigma,F_2}(b)$ and therefore $\frac{|\sigma(F_2)|}{|\sigma(F)|} * ne_{\sigma,F_1}(a) \geq \frac{|\sigma(F_1)|}{|\sigma(F)|} * ne_{\sigma,F_2}(b)$. Thus, $E \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}} E'$.

For $a \in A_2$ and $b \in A_1$, we can use the same reasoning as for $a \in A_1$ and $b \in A_2$.

"$\square = \mathsf{leximax}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E\cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}} E'\cap A_1$ and $E\cap A_2 \sqsupseteq_{F_2}^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}} E'\cap A_2$. So, we can use Lemma 4 to show $E \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}} E'$.

"$\Box = \mathsf{min}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{ne_\sigma,\mathsf{min}}} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{GCB}_{ne_\sigma,\mathsf{min}}} E' \cap A_2$.

Let $\mathsf{min}(ne_{\sigma,F}(E)) = \frac{|\{S \in \sigma(F_1) | a \in S\}| * |\sigma(F_2)| + |\{S \in \sigma(F_2) | a \in S\}| * |\sigma(F_1)|}{|\sigma(F)|}$ and $\mathsf{min}(ne_{\sigma,F}(E')) = \frac{|\{S' \in \sigma(F_1) | b \in S'\}| * |\sigma(F_2)| + |\{S' \in \sigma(F_2) | b \in S'\}| * |\sigma(F_1)|}{|\sigma(F)|}$.

If $a \in A_1$ and $b \in A_1$ then, we compare $|\{S \in \sigma(F_1) \mid a \in S\}| * |\sigma(F_2)|$ and $|\{S' \in \sigma(F_1) \mid b \in S'\}| * |\sigma(F_2)|$. $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{ne_\sigma,\mathsf{min}}} E' \cap A_1$ implies that $\mathsf{min}(ne_{\sigma,F_1}(E)) = \frac{|\{S \in \sigma(F_1) | a \in S\}|}{|\sigma(F_1)|} \geq \frac{|\{S' \in \sigma(F_1) | b \in S'\}|}{|\sigma(F_1)|} = \mathsf{min}(ne_{\sigma,F_1}(E'))$

If $a \in A_2$ and $b \in A_2$, we can use the same reasoning as above.

For $a \in A_1$ and $b \in A_2$ we have $\frac{|\sigma(F_1)|}{|\sigma(F)|} * ne_{\sigma,F_2}(b) \leq \frac{|\sigma(F_2)|}{|\sigma(F)|} * ne_{\sigma,F_1}(b')$ for every $b' \in E \cap A_1$. Let $ne_{\sigma,F_1}(b') = \mathsf{min}(ne_{\sigma,F_1}(E' \cap A_1))$, then $\frac{|\sigma(F_2)|}{|\sigma(F)|} * ne_{\sigma,F_1}(a) \geq \frac{|\sigma(F_1)|}{|\sigma(F)|} * ne_{\sigma,F_2}(b')$ and therefore $|\frac{|\sigma(F_2)|}{|\sigma(F)|} * ne_{\sigma,F_1}(a) \geq \frac{|\sigma(F_2)|}{|\sigma(F)|} * ne_{\sigma,F_1}(b)$. Thus, $E \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{min}}} E'$.

"$\Box = \mathsf{leximin}$": Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximin}}} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximin}}} E' \cap A_2$. So, we can use Lemma 5 to show $E \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximin}}} E'$. $\Box$

Proposition 82. *For* $\Box \in \{\mathsf{sum}, \mathsf{max}\}$ $\mathsf{GCB}_{ne_\sigma,\Box}$ *satisfies* weak reinstatement *and for* $\Box = \mathsf{leximax}$ *satisfies* strong reinstatement *with* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$.

Proof. Let $F = (A, R)$ be an AF, $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$ and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E), a \notin E$, and $a \notin (E^- \cup E^+)$.

"$\Box = \mathsf{sum}$": Since $ne_{\sigma,F}(a) \geq 0$, $\sum_{b \in E \cup \{a\}} ne_{\sigma,F}(b) \geq \sum_{c \in E} ne_{\sigma,F}(c)$ and therefore $E \cup \{a\} \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}} E$. If $ne_{\sigma,F}(a) = 0$ then $E \cup \{a\} \equiv_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{sum}}} E$, then *strong reinstatement* is violated.

"$\Box = \mathsf{max}$": If $\mathsf{max}(ne_{\sigma,F}(E \cup \{a\}) = ne_{\sigma,F}(a)$, then $ne_{\sigma,F}(a) \geq \mathsf{max}(ne_{\sigma,F}(E))$ and $E \cup \{a\} \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}} E$.

If $\mathsf{max}(ne_{\sigma,F}(E \cup \{a\}) \neq ne_{\sigma,F}(a)$, then $\mathsf{max}(ne_{\sigma,F}(E \cup \{a\}) = \mathsf{max}(ne_{\sigma,F}(E))$, therefore $E \cup \{a\} \equiv_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{max}}} E$.

"$\Box = \mathsf{leximax}$": Let $ne_{\sigma,F}(E) = \langle ne_{\sigma,F}(a_1), \ldots, ne_{\sigma,F}(a_n) \rangle$ in descending order, then $ne_{\sigma,F}(E \cup \{a\}) = \langle ne_{\sigma,F}(a_1), \ldots, ne_{\sigma,F}(a_{i-1}), ne_{\sigma,F}(a), ne_{\sigma,F}(a_i), \ldots, ne_{\sigma,F}(a_n) \rangle$. If we compare $ne_{\sigma,F}(E)$ and $ne_{\sigma,F}(E \cup \{a\})$ with $\mathsf{leximax}$, then in the first $i-1$ steps the same numbers are compared. We know that $a \geq a_i$, so $E \cup \{a\} \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}} E$. Since $|E \cup \{a\}| > |E|$, it holds that at some points $j \leq i$, $a_j > a_{j+1}$. Thus, $E \cup \{a\} \sqsupset_F^{\mathsf{GCB}_{ne_\sigma,\mathsf{leximax}}} E$. $\Box$

Proposition 83. $\mathsf{GCB}_{ne_{\mathsf{gr}},\Box}$ *for* $\Box \in \{\mathsf{min}, \mathsf{leximin}\}$ *satisfies* weak reinstatement.

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Suppose $a \in \mathfrak{F}_F(E), a \notin E$, and $a \notin (E^- \cup E^+)$. Since the grounded extension is unique, there can only be one extension, thus $ne_{\mathsf{gr},F}(b) \in \{0,1\}$ for every $b \in A$.

If $\mathsf{min}(ne_{\mathsf{gr},F}(E)) = 1$, then since $a \in \mathfrak{F}_F(E)$, $ne_{\mathsf{gr},F}(a) = 1$, hence $\mathsf{min}(ne_{\mathsf{gr},F}(E \cup \{a\})) = 1$ and therefore $\mathsf{min}(ne_{\mathsf{gr},F}(E \cup \{a\})) = \mathsf{min}(ne_{\mathsf{gr},F}(E))$. If $ne_{\mathsf{gr},F}(E) = 0$, then since $ne_{\mathsf{gr},F}(a) \geq 0$, $\mathsf{min}(ne_{\mathsf{gr},F}(E \cup \{a\})) = 0$ and therefore $\mathsf{min}(ne_{\mathsf{gr},F}(E \cup \{a\})) = \mathsf{min}(ne_{\mathsf{gr},F}(E))$. Implying, $E \cup \{a\} \equiv_F^{\mathsf{GCB}_{ne_{\mathsf{gr}},\square}} E$ for $\square \in \{\mathsf{min}, \mathsf{leximin}\}$. $\square$

Proposition 84. $\mathsf{GCB}_{ne_\sigma,\square}$ *satisfies* syntax independence *for* $\square \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.

Proof. Let $F = (A, R)$, $F' = (A', R')$ be AFs and $E, E' \subseteq A$. Assume isomorphism $\gamma : A \to A'$, then $ne_{\sigma,F}(E) = ne_{\sigma,F'}(\gamma(E))$ and therefore, if $E \sqsupseteq_F^{\mathsf{GCB}_{ne_\sigma,\square}} E'$, then $\gamma(E) \sqsupseteq_{F'}^{\mathsf{GCB}_{ne_\sigma,\square}} \gamma(E')$. $\square$

Proposition 85. Kelly_ρ *is reflexive and transitive.*

Proof. Let $F = (A, R)$ be an AF and ρ an argument-ranking.

"reflexive" By definition, if $E = E'$ then $E \equiv_F^{\mathsf{Kelly}_\rho} E'$.

"transitive" Consider $E, E', E'' \subseteq A$ such that $E \sqsupseteq_F^{\mathsf{Kelly}_\rho} E'$ and $E' \sqsupseteq_F^{\mathsf{Kelly}_\rho} E''$. Then every argument of E is ranked better than every argument of E', i. e. for all $a \in E$ and $b \in E'$, $a \succ_F^\rho b$. We also know that, every argument of E' is ranked better than every argument of E'', i. e. for all $b \in E'$ and $c \in E''$, $b \succ_F^\rho c$. Therefore, for all $a \in E$, $b \in E'$ and $c \in E''$: $a \succ_F^\rho b \succ_F^\rho c$ This means that $E \sqsupseteq_F^{\mathsf{Kelly}_\rho} E''$. $\square$

Proposition 86. Kelly_ρ *violates* weak reinstatement *for every argument-ranking semantics* ρ*.*

Proof. Let $F = (A, R)$ be an AF and $E \subseteq A$. Assume $a \in \mathfrak{F}_F(E)$, $a \notin E$, and $a \notin (E^- \cup E^+)$. Then for every argument-ranking semantics ρ, we get:

$$E \cup \{a\} \asymp_F^{\mathsf{Kelly}_\rho} E$$

Thus, Kelly_ρ violates *weak reinstatement*. $\square$

Proposition 87. *If* ρ *satisfies* Abs*, then* Kelly_ρ *satisfies* syntax independence.

Proof. Let $F = (A, R)$, $F' = (A', R')$ be AFs, $E, E' \subseteq A$, and ρ an argument-ranking semantics. Assume isomorphism $\gamma : A \to A'$. If ρ satisfies *Abs* and $a \succ_F^\rho b$, then $\gamma(a) \succ_{F'}^\rho \gamma(b)$ for every $a, b \in A$. Thus, *syntax independence* is satisfied. $\square$

Proposition 88. $\mathsf{Fishburn}_\rho$ *is reflexive and transitive.*

Proof. Let $F = (A, R)$ be an AF and ρ an argument-ranking.

"reflexive" By definition, if $E = E'$, then $E \equiv_F^{\mathsf{Fishburn}_\rho} E'$.

"transitive" Consider $E, E', E'' \subseteq A$ such that $E \sqsupseteq_F^{\mathsf{Fishburn}_\rho} E'$ and $E' \sqsupseteq_F^{\mathsf{Fishburn}_\rho} E''$. For all arguments $a \in E \setminus E'$ and $b \in E'$, $a \succ_F^\rho b$ and for all $b' \in E' \setminus E''$ and $c \in E''$, $b' \succ_F^\rho c$. This implies that $a \succ_F^\rho c$.

Assume $x \in E \cap E''$, if $x \in E'$ then $x \succ_F^\rho c$, since, $b \succ_F^\rho c'$ for $c' \in E'' \setminus E'$. Suppose $x \notin E'$, then it has to hold that $b \succ_F^\rho x$, however $x \in E \setminus E'$ and therefore it has to hold that $x \succ_F^\rho b$, so there is no $x \in E \cup E''$ such that $x \notin E'$. This means that $E \sqsupseteq_F^{\mathsf{Fishburn}_\rho} E''$. □

Proposition 89. *If ρ satisfies* Abs*, then* $\mathsf{Fishburn}_\rho$ *satisfies* syntax independence.

Proof. Let $F = (A, R)$, $F' = (A', R')$ be AFs, $E, E' \subseteq A$, and ρ an argument-ranking semantics. Assume isomorphism $\gamma : A \to A'$. If ρ satisfies *Abs* then if $a \succ_F^\rho b$ then $\gamma(a) \succ_{F'}^\rho \gamma(b)$ for every $a, b \in A$ and therefore *syntax independence* is satisfied. □

Proposition 90. Eli_ρ *is reflexive and transitive.*

Proof. Let $F = (A, R)$ be an AF and ρ an argument ranking.

"reflexive" By definition, if $E = E'$ then $E \equiv_F^{\mathsf{Eli}_\rho} E'$.

"transitive" Consider $E, E', E'' \subseteq A$ such that $E \sqsupseteq_F^{\mathsf{Eli}_\rho} E'$ and $E' \sqsupseteq_F^{\mathsf{Eli}_\rho} E''$. Then there exits an argument $b \in E'$ such that $a \succ_F^\rho b$ for all $a \in E$ and there exists an argument $c \in E''$ such that $b' \succ_F^\rho c$ for all $b' \in E'$. This especially means that $b \succ_F^\rho c$ and therefore, $a \succ_F^\rho c$ for all $a \in E$. This means that $E \sqsupseteq_F^{\mathsf{Eli}_\rho} E''$. □

Proposition 91. *If ρ satisfies* In*, then* Eli_ρ *satisfies* composition.

Proof. Let $F = F_1 \cup F_2 = (A_1, R_1) \cup (A_2, R_2)$ with $A_1 \cap A_2 = \emptyset$ and argument-ranking semantics ρ satisfying *In*. Assume $E, E' \subseteq A_1 \cup A_2$ with $E \cap A_1 \sqsupseteq_{F_1}^{\mathsf{Eli}_\rho} E' \cap A_1$ and $E \cap A_2 \sqsupseteq_{F_2}^{\mathsf{Eli}_\rho} E' \cap A_2$. Then there is an argument $b_1 \in E' \cap A_1$ such that for all $a_1 \in E \cap A_1$, $a_1 \succ_{F_1}^\rho b_1$ and for F_2, there is an argument $b_2 \in E' \cap A_2$ such that for all $a_2 \in E \cap A_2$, $a_2 \succ_{F_2}^\rho b_2$. W.l.o.g. assume $b_1 \succ_F^\rho b_2$, then because of *In*, $a_2 \succ_F b_2$ for all $a_2 \in E \cap A_2$. For all $a_1 \in E \cap A_1$, $a_1 \succ_F b_1$ and because of transitivity we have, $a_1 \succ_F b_2$. So, there is an argument $b \in E'$ such that for all arguments $a \in E$, $a \succ_F^\rho$ and therefore $E \sqsupseteq_F^{\mathsf{Eli}_\rho} E'$. □

Proposition 92. *If ρ satisfies* Abs*, then* Eli_ρ *satisfies* syntax independence.

Proof. Let $F = (A, R)$, $F' = (A', R')$ be AFs, $E, E' \subseteq A$, and ρ an argument-ranking semantics. Assume isomorphism $\gamma : A \to A'$. If ρ satisfies *Abs*, then $\rho(F) = \rho(\gamma(F))$ for every $a \in A$. Thus, *syntax independence* is satisfied. □

Proposition 93. *Let $F = (A, R)$ be an AF, $E, E' \subseteq A$, and ρ a total argument-ranking semantics, then if and only if $E \sqsupseteq_F^{\mathsf{Eli}_\rho} E'$ then $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\min}} E'$.*

Proof. Let $F = (A, R)$ be an AF, $E, E' \subseteq A$, and ρ a total argument-ranking semantics.

Assume $E \sqsupset_F^{\mathsf{Eli}_\rho} E'$, then there exists an argument $b \in E'$ such that for all $a \in E$, $a \succ_F^\rho b$. Meaning that $\min(\rho(E)) > b$ and therefore $\min(\rho(E)) > \min(\rho(E'))$.

Assume $E \sqsupset_F^{\mathsf{GCB}_{\rho,\min}} E'$, then $\min(\rho(E)) > \min(\rho(E'))$, so for every argument $a \in E$, $a \geq \min(\rho(E))$. Hence, for every argument in E there is at least one argument ranked worse ($\min(\rho(E'))$). Thus, $E \sqsupset_F^{\mathsf{Eli}_\rho} E'$. □

A.7 TECHNICAL PROOFS OF CHAPTER 9

Proposition 94. *A social ranking function that satisfies* Independence from the Worst Set *and* Pareto-efficiency *also satisfies* Dominating set.

Proof. Let $\mathcal{R}$ be a preorder on $\mathcal{P}(S)$ and let $x, y \in S$ be elements such that exists $X^d \in \mathcal{P}(S)$ with $x \in X^d$ and for all X' with $y \in X'$ then $(X^d, X') \in \mathcal{R}$ and $(X', X^d) \notin \mathcal{R}$. Furthermore, let $w := \text{rank}_{\mathcal{R}}(X^d) + 1$. Consider preorder $\mathcal{R}'$ defined as follows: For any sets $X, Y \in \mathcal{P}(S)$, $(X, Y) \in \mathcal{R}'$ if and only if $(X, Y) \in \mathcal{R}$ and either $\text{rank}_{\mathcal{R}}(X) < w$ or $\text{rank}_{\mathcal{R}}(Y) < w$. We claim that:

$$\max_{X \in \mathcal{P}(S)} (\text{rank}_{\mathcal{R}'}(X)) = w$$

First, we assume for the sake of contradiction that there is a set X with $\text{rank}_{\mathcal{R}'}(X) = w' > w$. By definition, there is a sequence $(X_1, X_2) \in \mathcal{R}'$, $(X_2, X_3) \in \mathcal{R}'$, $\ldots$, $(X_{w'}, X) \in \mathcal{R}'$. As every sequence of $\mathcal{R}'$ is also valid in $\mathcal{R}$, the same sequence exists for $\mathcal{R}$, i. e., $(X_1, X_2) \in \mathcal{R}$, $(X_2, X_3) \in \mathcal{R}$, ..., $(X_{w'}, X) \in \mathcal{R}$. However, this means $\text{rank}_{\mathcal{R}}(X_{w'}) \geq w' - 1 \geq w$ and $\text{rank}_{\mathcal{R}}(X) \geq w' > w$, which contradicts $(X_{w'}, X) \in \mathcal{R}'$.

To see that $\max_{X \in \mathcal{P}(S)}(\text{rank}_{\mathcal{R}'}(X)) \geq w$, we first observe that, since $\text{rank}_{\mathcal{R}}(X^d) = w - 1$ there is a sequence $(X_1, X_2) \in \mathcal{R}$, $(X_2, X_3) \in \mathcal{R}$, ..., $(X_{w-1}, X) \in \mathcal{R}$. As this sequence is maximal, $\text{rank}_{\mathcal{R}}(X_i) < w$ for all elements X_i of the sequence. Hence, the same sequence exists in $\mathcal{R}'$. Finally, as $(X^d, X') \in \mathcal{R}$ for all $X' \in \mathcal{P}(S)$ with $y \in S$, $(X^d, \{y\}) \in \mathcal{R}$ and $\text{rank}_{\mathcal{R}}(X^d) < w$, we also have $(X^d, \{y\}) \in \mathcal{R}'$. Therefore, $(X_1, X_2) \in \mathcal{R}', \ldots, (X_{w-1}, X) \in \mathcal{R}', (X, \{y\}) \in \mathcal{R}$ implies that $\text{rank}_{\mathcal{R}'}(\{y\}) \geq w$.

Next, we claim that $x \succ^{\xi_{\mathcal{R}'}} y$, for all social ranking functions ξ that satisfy Pareto-efficiency: By definition, $\text{rank}_{\mathcal{R}'}(X^d) = w - 1$. Furthermore, $((X^d \setminus \{x\}) \cup \{x\}, X^d \setminus \{x\} \cup \{y\}) \in \mathcal{R}'$, and thus $\text{rank}_{\mathcal{R}'}((X^d \setminus \{x\}) \cup \{y\}) > w - 1$. This shows that $\text{rank}_{\mathcal{R}'}(X^d) < \text{rank}_{\mathcal{R}'}((X^d \setminus \{x\}) \cup \{y\})$. On the other hand, there can be no Z such that $\text{rank}_{\mathcal{R}'}(Z \cup \{y\}) < \text{rank}_{\mathcal{R}'}(Z \cup \{x\})$. Since $(X^d, Z \cup \{y\}) \in \mathcal{R}$, $\text{rank}_{\mathcal{R}}(Z \cup \{y\}) \geq w$ and thus $\text{rank}_{\mathcal{R}'}(Z \cup \{y\}) \geq w$. Thus, the claim follows directly from $\max_{X \in \mathcal{P}(S)}(\text{rank}_{\mathcal{R}'}(X)) = w$

Finally, if ξ also satisfies Independence from the worst set, it follows that $x \succ^{\xi_{\mathcal{R}}} y$, as $\mathcal{R}$ is just a refinement of the worst set of $\mathcal{R}'$. □

Proposition 95. *The singleton social ranking function satisfies Independence from the worst set.*

Proof. Consider set S, preorder $\mathcal{R}$ over powerset $\mathcal{P}(S)$. Assume $x, y \in S$ such that $x \succ^{\mathsf{ST}_{\mathcal{R}}} y$, this means that $(\{x\}, \{y\}) \in \mathcal{R}$ and $(\{y\}, \{x\}) \notin \mathcal{R}$. Therefore, $\text{rank}_{\mathcal{R}}(\{x\}) < \text{rank}_{\mathcal{R}}(\{y\}) = w$. Let $\mathcal{R}'$ be another preorder on $\mathcal{P}(S)$ such that:

- $\text{rank}_{\mathcal{R}}(X) = \text{rank}_{\mathcal{R}'}(X)$ for all $X \in \mathcal{P}(S)$ such that $\text{rank}_{\mathcal{R}}(X) < w$.
- $\text{rank}_{\mathcal{R}'}(X) \geq w$ for all $X \in \mathcal{P}(S)$ such that $\text{rank}_{\mathcal{R}}(X) = w$

Then, $\text{rank}_{\mathcal{R}}(\{x\}) = \text{rank}_{\mathcal{R}'}(\{x\})$ and $\text{rank}_{\mathcal{R}}(\{y\}) \leq \text{rank}_{\mathcal{R}'}(\{y\})$ and this implies $(\{x\}, \{y\}) \in \mathcal{R}'$ and $(\{y\}, \{x\}) \notin \mathcal{R}'$, therefore $x \succ^{\mathsf{ST}_{\mathcal{R}'}} y$. $\square$

Proposition 96. *Lexicographic Excellence Operator satisfies Pareto-efficiency.*

Proof. Let S be a set and $\mathcal{R}$ a preorder over $\mathcal{P}(S)$. Consider $x, y \in S$ such that

1. $\text{rank}_{\mathcal{R}}(Z \cup \{x\}) \leq \text{rank}_{\mathcal{R}}(Z \cup \{y\})$ for all $Z \in \mathcal{P}(S)$ with $x, y \notin Z$;
2. $\text{rank}_{\mathcal{R}}(Z \cup \{x\}) < \text{rank}_{\mathcal{R}}(Z \cup \{y\})$ for at least one $Z \in \mathcal{P}(S)$ with $x, y \notin Z$.

Consider sets $Z_1, \ldots, Z_n \in \mathcal{P}(S)$ such that Condition 2 holds. Among these, take those $Z_1, \ldots, Z_m$ (with $m \leq n$) for which $\text{rank}_{\mathcal{R}}(Z_j \cup \{x\}) = k$ (with $1 \leq j \leq m$) is minimal. At this level in the preorder, we have that $\text{rank}_{\mathcal{R}}(Z \cup \{x\}) = \text{rank}_{\mathcal{R}}(Z \cup \{y\})$ for each $Z \neq Z_j$. Hence, for every $Z \cup \{x\}$ there is exactly one corresponding set $Z \cup \{y\}$, except for each $Z_j \cup \{x\}$, because $\text{rank}_{\mathcal{R}}(Z_j \cup \{y\}) > k$. Thus, for each $Z \in \mathcal{P}(S)$ with $x, y \notin Z$:

$$|\{Z \cup \{x\} \in \mathcal{P}(S) \mid \text{rank}_{\mathcal{R}}(Z \cup \{x\}) = k\}| > \\ |\{Z \cup \{y\} \in \mathcal{P}(S) \mid \text{rank}_{\mathcal{R}}(Z \cup \{y\}) = k\}|$$

At level k, there are more sets containing x than those containing y, i. e. $x_{k,\mathcal{R}} > y_{k,\mathcal{R}}$.

It remains to show that $x_{i,\mathcal{R}} = y_{i,\mathcal{R}}$ for all $i < k$. By construction, for all $i < k$ and $Z \in \mathcal{P}(S) \setminus \{x, y\}$, we know that $\text{rank}_{\mathcal{R}}(Z \cup \{x\}) = \text{rank}_{\mathcal{R}}(Z \cup \{y\})$. Hence, for each set containing x there is exactly one set containing y, thus $x_{i,\mathcal{R}} = y_{i,\mathcal{R}}$ and $x \succ^{\mathsf{lex\text{-}cel}_{\mathcal{R}}} y$. $\square$

Proposition 97. *Let $F = (A, R)$ be an AF, $a, b \in A$ and τ an extension ranking. If $a \succeq_F^{\mathsf{lex\text{-}cel}_\tau} b$, then $a \succeq_F^\tau b$.*

Proof. Let $F = (A, R)$ be an AF, $a, b \in A$, and τ an extension ranking. Assume $a \succeq_F^{\mathsf{lex\text{-}cel}_\tau} b$, then there is an k such that for all $i < k$, $a_{i,\tau} = b_{i,\tau}$ and $a_{k,\tau} \geq b_{k,\tau}$.

If $b_{j,\tau} \neq 0$ for $1 \leq j \leq k$, then there is one $Y \subseteq A$ with $\text{rank}_\tau(Y) = j$ and $b \in Y$. W.l.o.g let j be the smallest number such that $b_{j,\tau} \neq 0$. Then $Y \sqsupseteq_F^\tau X$ for all $X \subseteq A$ with $a \in X$, therefore $b \succeq_F^\tau a$. Since, $b_{j,\tau} \leq a_{j,\tau}$, there has to be an $X' \subseteq A$ with $\text{rank}_\tau(X') = j$ and $a \in X'$ such that $X \equiv_F^\tau Y$, so $a \succeq_F^\tau b$.

If $b_{j,\tau} = 0$ for all $j \in \{1, \ldots, k\}$ and $a_{k,\tau} > 0$, then there is at least one $X \subseteq A$ with $a \in X$ and $\text{rank}_\tau(X) = k$ such that $X \sqsupset_F^\tau Y$ for all $Y \subseteq A$ with $b \in Y$, and therefore $a \succ_F^\tau b$. $\square$

Proposition 98. *Let $F = (A, R)$ be an AF, τ an extension-ranking semantics satisfying* σ-generalisation *for extension semantics σ and ξ a social ranking function satisfying* Independence from the worst set *and* Pareto-efficiency. *Then the social ranking argument-ranking semantics ξ_τ satisfies* σ-C.

Proof. Let $F = (A, R)$ be an AF, τ an extension-ranking semantics satisfying *σ-generalisation* for extension semantics σ and ξ a social ranking function satisfying *Independence from the worst set* and *Pareto-efficiency*. Furthermore, let $a \in \mathsf{cred}_\sigma(F)$ and $b \in \mathsf{rej}_\sigma(F)$. Then there has to be a $E \subseteq A$ with $a \in E$ such that $E \in \sigma(F)$ and for all $E' \subseteq A$ with $b \in E'$, $E' \notin \sigma(F)$. It follows that $\text{rank}_\tau((E \setminus \{a\}) \cup \{a\}) = 1 < \text{rank}_\tau((E \setminus \{b\}) \cup \{b\})$. Also, there is no S such that $\text{rank}_\tau(S \cup \{b\}) < \text{rank}_\tau(S \cup \{a\})$, since, due to the fact that $w = \max_{E \subseteq A}(\text{rank}_\tau(E)) = 2$, this would imply that $\text{rank}_\tau(S \cup \{b\}) = 1$ and therefore $S \cup \{b\} \in \sigma(F)$, making b credulously accepted in F with respect to σ. As ξ satisfies Pareto-efficiency, we have $a \succ_F^{\xi_\tau} b$.

Since τ satisfies *σ-generalisation*, we know that $\text{rank}_{\sqsupseteq^\tau}(E) = 1$ if and only if $\text{rank}_\tau(E) = 1$. Hence, it follows from Independence from the worst set that $a \succ_F^{\xi_\tau} b$ implies $a \succ_F^{\xi_\tau} b$. $\square$

Proposition 99. *Let $F = (A, R)$ be an AF, τ an extension-ranking semantics satisfying* σ-generalisation *for extension semantics σ and ξ a social ranking function satisfying* Independence from the worst set *and* Pareto-efficiency. *Then the social ranking argument-ranking semantics ξ_τ satisfies* σ-sk-C.

Proof. Let $F = (A, R)$ be an AF, τ an extension-ranking semantics satisfying *σ-generalisation* for extension semantics σ and ξ a social ranking function satisfying *Independence from the worst set* and *Pareto-efficiency*.

We know that every extension-ranking semantics satisfying *σ-generalisation* is a refinement of least-discriminating extension-ranking semantics LD^σ with respect to σ such that $\mathtt{max}_{\mathsf{LD}^\sigma}(F) = \mathtt{max}_\tau(F)$. Thus, it is enough to show this statement for LD^σ. Furthermore, consider $a, b \in A$ such that $a \in \mathsf{skep}_\sigma(F)$ and $b \notin \mathsf{skep}_\sigma(F)$. Assume there exists a $Z \subseteq A \setminus \{a, b\}$ such that $\text{rank}_\tau(Z \cup \{b\}) < \text{rank}_\tau(Z \cup \{a\})$. Since LD^σ has only two levels, this implies that $Z \cup \{b\} \in \mathtt{max}_{\mathsf{LD}^\sigma}(F)$ and thus $Z \cup \{b\} \in \sigma(F)$. As $a \in \mathsf{skep}_\sigma(F)$, we must have $a \in Z \cup \{b\}$. However, $a \notin Z$ and therefore $a \notin Z \cup \{b\}$. Hence, there is a contradiction and therefore such a Z cannot exists.

Since $b \notin \mathsf{skep}_\sigma(F)$, there exists a set $X \subseteq A$ such that $X \in \mathtt{max}_{\mathsf{LD}^\sigma}(F)$ and $b \notin X$. Because $a \in \mathsf{skep}_\sigma(F)$, $(X \setminus \{a\}) \cup \{b\} \notin \mathtt{max}_{\mathsf{LD}^\sigma}(F)$. Consequently, Pareto-efficiency implies $a \succ_F^{\xi_{\mathsf{LD}^\sigma}} b$.

As $a \succ_F^{\xi_{\mathsf{LD}^\sigma}} b$ holds for LD^σ and τ is a refinement of LD^σ, it follows from Independence from the worst set that the same holds for τ, i. e., $a \succ_F^{\xi_\tau} b$. $\square$

Proposition 100. *Let ξ be a social ranking such that $\xi_{\mathsf{r\text{-}ad}}$ satisfies* ad-C. *Then ξ satisfies* Dominating set *for all* r-ad*-realisable preorders $\sqsupseteq$.*

Proof. Let $\sqsupseteq$ be a r-ad-realisable preorder and $F = (A, R)$ be an AF that induces $\sqsupseteq$. Assume $x, y \in A$ such that there exists $X \subseteq A$ with $x \in X$ for which we have $X \sqsupset Y$ for all Y with $y \in Y$. Let ξ be a social ranking function such that $\xi_\sqsupseteq$ satisfies ad-*C*.

Assume that the set X is not admissible. That means one of the following two cases must apply:

$$(1) \quad \mathsf{Conflicts}_F(X) \neq \emptyset \quad \text{or,} \quad (2) \quad \mathsf{Undef}_F(X) \neq \emptyset.$$

to (1)**:** Then, there is some attack $(a, b) \in \mathsf{Conflicts}_F(X)$ for $a, b \in X$. From $X \sqsupset Y$ it follows that $\mathsf{Conflicts}_F(X) \subseteq \mathsf{Conflicts}_F(Y)$ and thus $(a, b) \in \mathsf{Conflicts}_F(Y)$. Now, if $y = a$ or $y = b$ it follows that $y \in X$ which directly contradicts our assumption, because of $X \equiv Y'$ for $Y' = X$ with $y \in Y'$. However, if $y \neq a$ and $y \neq b$, we can construct $Y' = Y \setminus \{a, b\}$. Clearly, that means we either have $\mathsf{Conflicts}_F(Y') = \emptyset$ which means $Y \sqsupset X$ or we have $\mathsf{Conflicts}_F(Y') \neq \emptyset$ which implies $X \asymp Y'$. Because of $y \in Y'$ both cases contradict the initial assumption, hence we must have that $\mathsf{Conflicts}_F(X) = \emptyset$, i. e. the set X is conflict-free.

to (2)**:** Then, there exists an argument $a \in \mathsf{Undef}_F(X)$ which is not defended by X. Consider now the set $Y' = \{y\}$ for which we either have that $\mathsf{Undef}_F(Y') = \emptyset$ or $\mathsf{Undef}_F(Y') = \{y\}$. If $\mathsf{Undef}_F(Y') = \emptyset$, it follows directly that $Y' \sqsupset X$, contradicting our initial assumption. On the other hand, for $\mathsf{Undef}_F(Y') = \{y\}$ we distinguish between two cases:

$$(2.1) \quad y = x, \qquad (2.2) \quad y \neq x$$

Clearly, if $x = y$ we contradict our initial assumption because $X \equiv Y''$ for $Y'' = X$. Consider now the case $y \neq x$. That means, we have that $\mathsf{Undef}_F(X) \asymp \mathsf{Undef}_F(Y')$ and thus $X \asymp Y'$. Thus, it follows that $\mathsf{Undef}_F(X) = \emptyset$, i. e. X defends all its elements.

That means X is admissible and thus it follows directly that $x \in \mathsf{cred}_{\mathsf{ad}}(F)$.

From $\mathsf{Undef}_F(X) = \emptyset$ and $X \sqsupset_F^{UD} Y$ for all Y it follows that $\mathsf{Undef}_F(Y) \neq \emptyset$. Since $\sqsupseteq$ satisfies ad-*generalisation* it follows that $Y \notin \mathsf{ad}(F)$ for all Y and thus also $y \in \mathsf{rej}_{ad}(F)$. Consequently, it follows from ad-*C* that $x \succ^{\xi_\sqsupseteq} y$. Hence, Dominating set is satisfied. □

Proposition 101. $\mathsf{lex\text{-}cel}_\tau$ *satisfies* Abs *for* $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Proof. Let $F = (A, R)$ and $F' = (A', R')$ be AFs such that there is a isomorphism $\gamma : F \to F'$. Consider $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$. Assume $a \succeq^{\mathsf{lex\text{-}cel}_\tau} b$, then there exists a k with $0 \leq k \leq |\mathcal{P}(A)|$ such that $a_{k,\tau} \geq b_{k,\tau}$ and for $i < k$, $a_{i,\tau} = b_{i,\tau}$. Since τ satisfies *syntax independence*, for every pair of sets $E, E' \in A$ with $E \sqsupseteq_F^\tau E'$, $\gamma(E) \sqsupseteq_{F'}^\tau \gamma(E')$. Also if $a \in E$ then $\gamma(a) \in \gamma(E)$. So, the number of sets containing a in F on rank k with respect to τ is the same as the number of sets containing $\gamma(a)$ in F' on rank k with respect to τ. Hence, if $a \succeq_F^\tau b$ then $\gamma(a)_{k,\tau} > \gamma(b)_{k,\tau}$ and therefore $\gamma(a) \succeq_{F'}^\tau b$. □

Proposition 102. $\mathsf{lex\text{-}cel}_\tau$ *satisfies* In *for* $\tau \in \{\mathsf{r\text{-}ad}, \mathsf{r\text{-}co}, \mathsf{r\text{-}gr}, \mathsf{r\text{-}pr}, \mathsf{r\text{-}sst}\}$.

Proof. Let $F = (A, R)$ be an AF, $F' = (A', R') \in cc(F)$ and $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-gr}, \text{r-pr}, \text{r-sst}\}$. Assume $a, b \in A'$ such that $a \succeq_F^{\text{lex-cel}_\tau} b$ then there exists a $k \in \mathbb{N}$ such that $a_{k,\sqsupseteq_F^\tau} > b_{k,\sqsupseteq_F^\tau}$ and for all $i < k$: $a_{k,\sqsupseteq_F^\tau} = b_{k,\sqsupseteq_F^\tau}$.

Let $E, E^* \subseteq A$ be two sets such that $E \sqsupseteq_F^\tau E^*$. Suppose $F' \cup (A \setminus A', R \setminus, R') = F$ and since $F' \in cc(F)$, we know that $F' \cap (A \setminus A', R \setminus, R') = \emptyset$. So, *decomposition* implies $E \cap A' \sqsupseteq_{F'}^\tau E^* \cap A'$ and therefore we have $\text{rank}_{\sqsupseteq_{F'}^\tau}(E \cap A') \leq \text{rank}_{\sqsupseteq_{F'}^\tau}(E^* \cap A')$. If $a \in E$ then $a \in E \cap A'$, the number of highly ranked sets containing a can only grow:

$$a_{1,\sqsupseteq_F^\tau} \leq a_{1,\sqsupseteq_{F'}^\tau}$$

Next, we examine two cases for k:

$a_{k,\sqsupseteq_{F'}^\tau} \leq b_{k,\sqsupseteq_{F'}^\tau}$: This case implies that there has to be a $k' < k$ such that:

$$a_{k,\sqsupseteq_{F'}^\tau} > b_{k,\sqsupseteq_{F'}^\tau} \quad \text{hence} \quad a \succeq_{F'}^{\text{lex-cel}_\tau} b$$

$a_{k,\sqsupseteq_{F'}^\tau} > b_{k,\sqsupseteq_{F'}^\tau}$: For this case, we have for all $i < k$:

$$a_{i,\sqsupseteq_{F'}^\tau} \geq b_{i,\sqsupseteq_{F'}^\tau} \quad \text{implying} \quad a \succeq_{F'}^{\text{lex-cel}_\tau} b$$

□

Proposition 103. *Let $F = (A, R)$ be an AF, if extension-ranking semantics τ satisfies* respect conflicts *and social ranking function ξ satisfies* Dominating set*, then ξ_τ satisfies* SC.

Proof. Let $F = (A, R)$ be an AF and $a, b \in A$, $(b, b) \in R$ and $(a, a) \notin R$, then $\{a\} \in \text{cf}(F)$ and for all $E' \subseteq A$ with $b \in E'$ it holds that $E' \notin \text{cf}(F)$. Because of *respect conflicts* we have $\{a\} \sqsupset E'$ and therefore because of Dominating set, we have $a \succ_F^{\xi\tau} b$. □

Proposition 104. lex-cel_τ *satisfies* AvsFD *for* $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-gr}, \text{r-pr}, \text{r-sst}\}$.

Proof. Let $F = (A, R)$ be an acyclic AF and τ an extension-ranking semantics with $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-gr}, \text{r-pr}, \text{r-sst}\}$. Assume for arguments $a, b \in A$ every path $P(u, a)$ in F from unattacked u to a has $l_P = 0 \ mod \ 2$ and there exists unattacked $v \in b_F^-$. Since F is acyclic and every path from an unattacked argument u to a is even, we know that u is an indirect defender of a. In particular this also means that $a \in \text{gr}(F)$ and therefore $a \in \text{cred}_{\text{ad}}(F)$. On the other hand b is not admissible since the attack from v is not counterattacked implying $b \in \text{rej}_{\text{ad}}(F)$. Since lex-cel_τ satisfies σ-*C* for $\sigma \in \{\text{ad}, \text{co}, \text{gr}, \text{pr}, \text{sst}\}$ this implies that $a \succeq_F^{\text{lex-cel}_\tau} b$ and thus *AvsFD* is satisfied. □

Proposition 105. lex-cel_τ *satisfies* sσ-S *for* $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-pr}, \text{r-sst}\}$ *and* $\sigma \in \{\text{ad}, \text{co}, \text{pr}, \text{sst}\}$.

Proof. Let $F = (A, R)$ be an AF, τ an extension-ranking semantics with $\tau \in \{\text{r-ad}, \text{r-co}, \text{r-pr}, \text{r-sst}\}$ and extension semantics $\sigma \in \{\text{ad}, \text{co}, \text{pr}, \text{sst}\}$. Assume arguments $a, b \in A$, where a strongly σ supports b. Then for every set $E \in \mathtt{max}_{\sqsupseteq^\tau}(F)$ with $b \in E$ there is at least one set E' with $a \in E'$ and $b \notin E'$ such that $E' \subseteq E$. This implies $|a_{0,\sqsupseteq^\tau}| > |b_{0,\sqsupseteq^\tau}|$, since $a \in E$ and $b \notin E'$. Thus, $a \succeq_F^{\text{lex-cel}_\tau} b$. □

Proposition 106. $\text{lex-cel}_{\text{r-co}}$ *satisfies* UR.

Proof. Let $\{x_1, \dots, x_n\}$ be a set of arguments, and $\succ$ be a preorder such that $\succ$: $x_1 \succ \dots \succ x_n$. To show that $\text{lex-cel}_{\text{r-co}}$ satisfies *UR*, we present a construction of an AF that induces $\succ$ with respect to $\text{lex-cel}_{\text{r-co}}$. For the constructed AF F, the following statement holds:

$$ne_{\text{co},F}(x_1) > \dots > ne_{\text{co},F}(x_n) \tag{A.1}$$

Equation A.1 implies that $x_1\, {}_{1,\sqsupseteq_F^{\text{r-co}}} > \dots > x_n\, {}_{1,\sqsupseteq_F^{\text{r-co}}}$ and therefore:

$$x_1 \succ_F^{\text{lex-cel}_{\text{r-co}}} \dots \succ_F^{\text{lex-cel}_{\text{r-co}}} x_n$$

We begin with the construction for n being an even number. Consider AF $F = (\{x_1, \dots, x_n\}, R)$ with:

$$\begin{aligned} R = \{&(x_1, x_n), (x_n, x_1), (x_n, x_n) \\ &(x_3, x_{n-2}), (x_{n-2}, x_3), (x_3, x_n) \\ &\dots \\ &(x_{2i+1}, x_{n-2i}), (x_{n-2i}, x_{2i+1}), (x_{2i+1}, x_{n-2(i+1)}) \\ &\dots \\ &(x_{n-1}, x_2), (x_2, x_{n-1}), (x_{n-1}, x_4)\} \end{aligned}$$

An example AF for $n = 8$ is depicted in Figure A.4.

Since x_n is self-attacking, we have $ne_{\text{co},F}(x_n) = 0$. Next, we present a function to compute $ne_{\text{co},F}(x_i)$ for the remaining arguments x_i.

If i is odd: If x_i is in a complete extension, then every argument x_j with $j < i$ with $j \mod 2 = 1$ is also in the complete extension, as x_j is defended by x_i. Thus, for every complete extension E we have $\{x_1, \dots, x_j, \dots, x_i\} \subseteq E$, where $j < i$ with $j \mod 2 = 1$.

For every x_k with $k \geq i$ with $k \mod 2 = 1$, there is one complete extension containing only odd numbered arguments. In our example for $n = 8$, consider $i = 3$, then argument x_1 is defended by x_3, thus $\{x_1, x_3\}$ is a complete extension. Argument x_5 can be added to this set and we receive another complete extension, $\{x_1, x_3, x_5\}$. Argument x_7 can be added to the previous set for a complete extension. Thus, for $i = 3$, we have three complete extension containing only odd numbered arguments. For any x_i there are $\lceil \frac{n-i}{2} \rceil$ complete

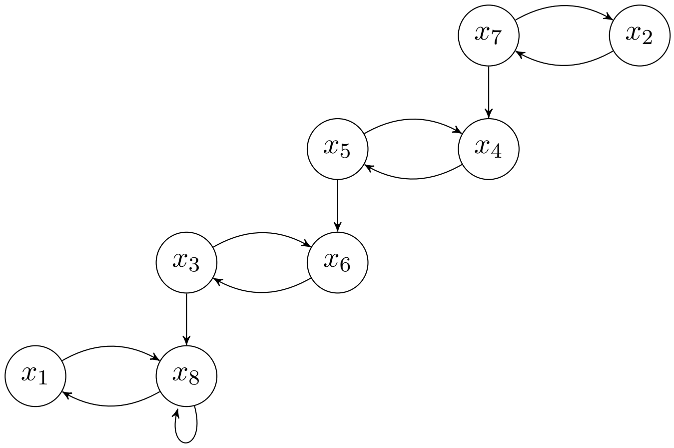

Figure A.4: Constructed AF F for $n = 8$.

extensions containing only odd numbered arguments, since between i and n there are $\lceil \frac{n-i}{2} \rceil$ odd numbers. We call these complete extensions *odd numbered complete extensions of* x_i.

For every even numbered argument with an index l smaller than $l < n-i$ there is an complete extension containing x_i and x_l. For $n = 8$ and $i = 3$, the set $\{x_1, x_3, x_2\}$ is a complete extension. For x_4, we need to add the defender of x_4 into our set, so x_2. Thus, $\{x_1, x_3, x_2, x_4\}$ is a complete extension. Formally, these complete extensions are of the form $\{x_1, \dots, x_i, x_l, \dots, x_o, \dots, x_2\}$, where $o \leq l < n - i$ with $o, l \mod 2 = 0$. Since there are $\lfloor \frac{n-i}{2} \rfloor$ even numbers between l and 2, there are $\lfloor \frac{n-i}{2} \rfloor$ complete extensions of this form. We call these complete extensions *even numbered complete extensions of* x_i.

The remaining extensions are combinations of the two cases above. For an odd numbered argument x_k with a bigger index than $k > i$, there are extensions containing even arguments x_l with a smaller index than $l < n - k$. In our example for $n = 8$ and $i = 3$, the set $\{x_1, x_3, x_5, x_2\}$ is a complete extension. These are the even numbered complete extensions of x_k. Thus, we get:

$$\lfloor \frac{n-i-2}{2} \rfloor + \lfloor \frac{n-i-4}{2} \rfloor + \dots + \lfloor \frac{n-(n-1)}{2} \rfloor$$

If we examine this sequences closer, we see that $\lfloor \frac{n-(n-1)}{2} \rfloor = 0$, $\lfloor \frac{n-(n-3)}{2} \rfloor = 1$, $\lfloor \frac{n-(n-5)}{2} \rfloor = 2$, $\lfloor \frac{n-(n-7)}{2} \rfloor = 3$, and so on. Thus, we can characterise this sequence with the triangular numbers. Consequently, there are $\frac{\lfloor \frac{n-i-2}{2} \rfloor^2 + \lfloor \frac{n-i-2}{2} \rfloor}{2}$ such complete extension. We call these the *mixed complete extensions of* x_i.

Combing these three observations, we get:

$$\begin{aligned} ne_{\mathsf{co},F}(x_i) &= \lceil \frac{n-i}{2} \rceil + \lfloor \frac{n-i}{2} \rfloor + \frac{\lfloor \frac{n-i-2}{2} \rfloor^2 + \lfloor \frac{n-i-2}{2} \rfloor}{2} \\ &= n - i + \frac{\lfloor \frac{n-i-2}{2} \rfloor^2 + \lfloor \frac{n-i-2}{2} \rfloor}{2} \end{aligned}$$

i **is even:** Since x_i is attacked, any complete extension E containing x_i must contain the recursive defenders of x_i, so $\{x_2, \dots, x_j, \dots, x_i\} \subseteq E$ for $j < i$ with $j \bmod 2 = 0$.

For every even argument x_k with index $k \geq i$ (except n), there are two complete extensions $\{x_2, \dots, x_i, \dots, x_k\}$ and $\{x_2, \dots, x_i, \dots, x_k, x_1\}$, where every argument except x_1 has an even index. The arguments between x_i and x_k are need to ensure that x_k is defended. For our example for $n = 8$ and $i = 4$, we have four complete extensions, $\{x_2, x_4\}$, $\{x_2, x_4, x_1\}$, $\{x_2, x_4, x_6\}$, and $\{x_2, x_4, x_6, x_1\}$. Thus, there are $n - i$ such complete extensions.

The remaining complete extensions can contain more than one odd numbered argument. For every odd numbered argument with an index l such that $l < n - i$, we find complete extensions containing x_i and x_l. These are the mixed complete extensions of x_l. Thus, we can characterise the number of these complete extension containing x_i with the triangular numbers.

Consequently, we get:

$$ne_{\mathsf{co},F}(x_i) = n - i + \frac{\frac{n-i-2}{2}^2 + \frac{n-i-2}{2}}{2}$$

Next, we show that $ne_{\mathsf{co},F}(x_i) > ne_{\mathsf{co},F}(x_{i+1})$. If i is even, then:

$$\frac{\frac{n-i-2}{2}^2 + \frac{n-i-2}{2}}{2} > \frac{\lfloor \frac{n-i-1}{2} \rfloor^2 + \lfloor \frac{n-i-1}{2} \rfloor}{2}$$

So, $ne_{\mathsf{co},F}(x_i) > ne_{\mathsf{co},F}(x_{i+1})$.

If i is odd, then:

$$\frac{\lfloor \frac{n-i-2}{2} \rfloor^2 + \lfloor \frac{n-i-2}{2} \rfloor}{2} = \frac{\frac{n-i-1}{2}^2 + \frac{n-i-1}{2}}{2}$$

Implying, $ne_{\mathsf{co},F}(x_i) > ne_{\mathsf{co},F}(x_{i+1})$.

For n being an odd number, consider the AF $F' = (\{x_1, \dots, x_n\}, R)$ with:

$$\begin{aligned} R = \{&(x_2, x_n), (x_n, x_2), (x_n, x_n) \\ &(x_4, x_{n-2}), (x_{n-2}, x_4), (x_3, x_n) \\ &\dots \\ &(x_{2i+1}, x_{n-2i}), (x_{n-2i}, x_{2i+1}), (x_{2i+1}, x_{n-2(i+1)}) \\ &\dots \\ &(x_{n-1}, x_3), (x_3, x_{n-1}), (x_{n-1}, x_5)\} \end{aligned}$$

The difference between the two constructions is that F' contains one unattacked argument x_1. The structure of the remaining arguments is isomorphic to F. Since argument x_1 is unattacked, it is in every complete extension, and $\{x_1\} \in \mathsf{co}(F')$. Thus, $ne_{\mathsf{co},F'}(x_1) > ne_{\mathsf{co},F'}(x_i)$ for $1 < i \leq n$. For the remaining arguments $x_2, \ldots, x_n$, the same reasoning as for F applies, with the odd and even cases flipped.

Consequently, Equation A.1 holds for every n and therefore, $\mathsf{lex\text{-}cel}_{\mathsf{r\text{-}co}}$ satisfies *UR*. $\square$

A.8 TECHNICAL PROOFS OF CHAPTER 10

Proposition 107. LD^σ EXTENSION RANKING *is in* **P** *for* $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{st}\}$.

Proof. Consider AF $F = (A, R)$, $E, E' \subseteq A$, and extension semantics $\sigma \in \{\mathsf{cf}, \mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{st}\}$. In order to decide $E \sqsupseteq_F^{\mathsf{LD}^\sigma} E'$, we first check if $E \in \sigma(F)$ and $E' \in \sigma(F)$. If E is a σ-extension, then $E \in \mathtt{max}_{\mathsf{LD}^\sigma}(F)$, $E \sqsupseteq_F^{\mathsf{LD}^\sigma} E''$ for every $E'' \subseteq A$. So, LD^σ EXTENSION RANKING would return `Yes` for E and E'.

If $E \notin \sigma(F)$ and $E' \notin \sigma(F)$, then $E \equiv_F^{\mathsf{LD}^\sigma} E'$, hence LD^σ EXTENSION RANKING returns `Yes` for E and E'.

The only case missing is $E \notin \sigma(F)$ and $E' \in \sigma(F)$, so $E' \in \mathtt{max}_{\mathsf{LD}^\sigma}(F)$ and therefore also $E' \sqsupset_F^{\mathsf{LD}^\sigma} E$. Thus, LD^σ EXTENSION RANKING returns `No` for E and E'.

The upper bound of the complexity of LD^σ EXTENSION RANKING boils down to the complexity of deciding if $E \in \sigma(F)$, and this problem is the verification problem VER_σ and solvable with an polynomial time algorithm. $\square$

Proposition 108. LD^σ EXTENSION RANKING *is* **coDP***-complete for* $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$.

Proof. Consider AF $F = (A, R)$, $E, E' \subseteq A$, and extension semantics $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$. In order to show that $E \sqsupseteq_F^{\mathsf{LD}^\sigma} E'$ holds it is enough to check if $E \in \sigma(F)$ or $E' \notin \sigma(F)$ holds. The problem of verifying if a set of arguments is a σ-extension for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$ is **coNP**-complete.

We check the complementary problem to LD^σ EXTENSION RANKING in the following way. We check if $E \notin \sigma(F)$ holds and if $E' \in \sigma(F)$ holds, the first check is possible in **NP** and the second part is possible in **coNP**. If both E and E' return `Yes` for their corresponding check, then the complementary problem returns `Yes` as well. So, the complementary problem to LD^σ EXTENSION RANKING is in **DP** and therefore LD^σ EXTENSION RANKING is in **coDP**.

Next, we present a reduction from SAT-UNSAT to the complementary problem of LD^σ EXTENSION RANKING based on the *standard translation* by Dvorák and Dunne (2017). Consider two propositional formulae φ and ψ in CNF given by the set of clauses $C = C_\varphi \cup C_\psi$ over atoms $Z = X \cup Y$ with $X = \{x_1, \ldots, x_n\}$ and $Y =$

$\{y_1, \ldots, y_m\}$. We define the corresponding AF $F_{(\varphi,\psi)} = (A, R)$ as follows:

$$
\begin{aligned}
A = &\{a, b, \varphi, \bar{\varphi}, \psi, \bar{\psi}\} \cup C \cup Z \cup \bar{Z} \\
R = &\{(x, \bar{x}), (\bar{x}, x) \mid x \in X\} \cup \{(y, \bar{y}), (\bar{y}, y) \mid y \in Y\} \cup \\
&\{(x, c) \mid x \in c, c \in C\} \cup \{(\bar{x}, c) \mid \bar{x} \in c, c \in C\} \cup \\
&\{(y, c) \mid y \in c, c \in C\} \cup \{(\bar{y}, c) \mid \bar{y} \in c, c \in C\} \cup \\
&\{(c, \varphi) \mid c \in C_\varphi\} \cup \{(c, \psi) \mid c \in C_\psi\} \cup \\
&\{(c, c) \mid c \in C\} \cup \\
&\{(\bar{\varphi}, x) \mid x \in X\} \cup \{(\bar{\varphi}, \bar{x}) \mid x \in X\} \cup \\
&\{(\bar{\psi}, y) \mid y \in Y\} \cup \{(\bar{\psi}, \bar{y}) \mid y \in Y\} \cup \\
&\{(\psi, \bar{\psi}), (\varphi, \bar{\varphi})\} \cup \\
&\{(a, x_\varphi) \mid x_\varphi \in \{\varphi, \bar{\varphi}\} \cup X \cup \bar{X} \cup C_\varphi\}\} \cup \\
&\{(b, y_\psi) \mid y_\psi \in \{\psi, \bar{\psi}\} \cup Y \cup \bar{Y} \cup C_\psi\}\} \\
&\{(a, b), (b, a)\}
\end{aligned}
$$

where X contains the atoms of φ, Y contains the atoms of ψ. With $\bar{Z}$ we denote the negations of the atoms Z. An example reduction for $\varphi = (x_1 \vee x_2 \vee \bar{x_3}) \wedge (\bar{x_1} \vee \bar{x_2} \vee \bar{x_3})$ and $\psi = y_1 \wedge \bar{y_1}$ is depicted in Figure A.5.

We show that if $I(X, Y)$ is a `Yes`-instance of SAT-UNSAT, then $F_{(\varphi,\psi)}$ with $E' = \{a\}$ and $E = \{b\}$ is a `Yes`-instance of the complementary problem of LD$^\sigma$ EXTENSION RANKING for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$. If $I(X, Y)$ is a `Yes`-instance of SAT-UNSAT, then φ is satisfiable and ψ is unsatisfiable. Thus, there is an assignment $I(X)$ of X such that every clause of φ is satisfied. Dvorák and Dunne (2017) have shown that this assignment induces a σ-extension for the φ part, i. e. $S \cup \{\varphi\}$ is a σ-extension, where $S = \{x_i \mid I(x_i) = true\} \cup \{\bar{x_i} \mid I(x_i) = false\}$ for $F_\varphi = (A_\varphi, R \cap (A_\varphi \times A_\varphi))$ where $A_\varphi = \{X \cup \bar{X} \cup C_\varphi \cup \{\varphi, \bar{\varphi}\}\}$. Since b attacks every argument in $A \setminus A_\varphi \cup \{b\}$, we get a σ-extension for $F_{(\varphi,\psi)}$ by adding b to $S \cup \{\varphi\}$. Thus, $E = \{b\}$ cannot be a σ-extension of $F_{(\varphi,\psi)}$ if φ is satisfiable.

If ψ is unsatisfiable, then Dvorák and Dunne (2017) have shown that there is no non-empty σ-extension in $F_\psi = (A_\psi, R \cap (A_\psi \times A_\psi))$ where $A_\psi = \{Y \cup \bar{Y} \cup C_\psi \cup \{\psi, \bar{\psi}\}\}$. For every possible assignment $I(Y)$ of Y there is at least one clause of ψ not satisfied, this implies that for every possible set $S' = \{y_i \mid I(y_i) = true\} \cup \{\bar{y_i} \mid I(y_i) = false\}$ induced by these assignments there is at least on clause argument $c_i \in C_\psi$ not attacked by S', thus ψ cannot be defended and therefore $\bar{\psi}$ is not defeated. Hence, none of the arguments out of $Y \cup \bar{Y}$ can be part of any σ-extension. For $\bar{\psi}$ to be part of any σ-extension, ψ has to be defeated, which is not possible, since every attacker of ψ is self-attacking. Thus, $\emptyset$ is the only σ-extension of F_ψ. Argument a attacks every argument in $A \setminus A_\psi \cup \{a\}$, hence $\{a\}$ is a σ-extension if ψ is unsatisfiable. So, $E' \in \sigma(F_{(\varphi,\psi)})$ and $E \notin \sigma(F_{(\varphi,\psi)})$ if $I(X, Y)$ is a `Yes`-instance of SAT-UNSAT.

If $I(X, Y)$ is a `No`-instance of SAT-UNSAT, then either φ is unsatisfiable or ψ is satisfiable. If φ is unsatisfable, then $\{b\}$ is a σ-extension of $F_{(\varphi,\psi)}$, to show this we follow the reasoning of the ψ case from above. Thus, $F_{(\varphi,\psi)}$ with $E' = \{a\}$ and

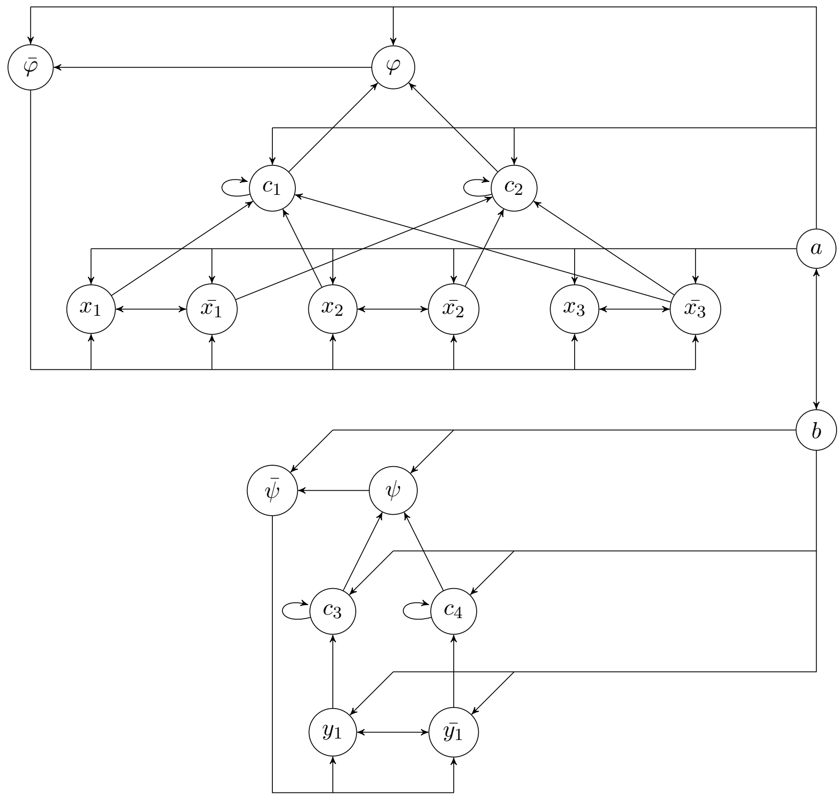

Figure A.5: Example reduction for **coDP**-hardness of LD$^\sigma$ Extension Ranking. $F_{(\varphi,\psi)}$ for propositional formulae $\varphi = (x_1 \lor x_2 \lor \bar{x_3}) \land (\bar{x_1} \lor \bar{x_2} \lor \bar{x_3})$ and $\psi = y_1 \land \bar{y_1}$.

$E = \{b\}$ is a No-instance of the complementary problem of LD$^\sigma$ Extension Ranking for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$. If ψ is satisfiable, then $\{a\}$ is not a σ-extension of $F_{(\varphi,\psi)}$, to show this we use the reasoning for the φ case from above. Thus, $F_{(\varphi,\psi)}$ with $E' = \{a\}$ and $E = \{b\}$ is a No-instance of the complementary problem of LD$^\sigma$ Extension Ranking for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$.

We have shown that the complementary problem to LD$^\sigma$ Extension Ranking is **DP**-complete, thus LD$^\sigma$ Extension Ranking is **coDP**-complete. □

Proposition 109. $\sqsupseteq^\tau$ Extension Ranking *for* $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}, \mathsf{c\text{-}Conflicts}, \mathsf{c\text{-}Undef}, \mathsf{c\text{-}Condef}, \mathsf{c\text{-}Unatt}\}$ *is in* **P**.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. We describe an polynomial-time algorithm to calculate each of underlying base relation.

"Conflicts": To calculate $\mathsf{Conflicts}_F(E)$, we need to check for every two arguments $a, b \in E$ if they share an attack. This is possible in polynomial time. We store the AF F as an adjacency matrix and then check for neighbourhood of these arguments. This check is possible in constant time. Thus, we have to do this look-up at most $|E|^2$ times.

"Undef": For $\mathsf{Undef}_F(E)$, we evalute the characteristic function $\mathfrak{F}_F(E)$. Dunne and Wooldridge (2009) have shown that this is possible by a deterministic polynomial-time algorithm.

"Unatt": For $\mathsf{Unatt}_F(E)$, we use the adjacency matrix representation of F. To compute a_F^+ for $a \in E$, we look at the corresponding column of a in the adjacency matrix and append all found arguments onto our set. To compute E_F^+, we repeat this process $|E|$ times. Thus, we can compute E_F^+ by a deterministic polynomial-time algorithm.

"Condef": We start with $\mathfrak{F}^*_{1,F}(E) = E$. Then in every iteration i, we add at least one argument into $\mathfrak{F}^*_{i+1,F}(E)$ or stop. We have $E \subseteq \mathfrak{F}^*_F(E) \subseteq A$. Thus, after at most $|A|$ steps we reach a fixed point as shown in Proposition 10. Thus, to compute $\mathsf{Condef}_F(E)$ we need at most $|A|$ steps and therefore it is possible by a deterministic polynomial-time algorithm.

For each of the base relation $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$, we can compute $\tau_F(E)$ in polynomial time. Thus, it is clear that checking if $\tau_F(E) \subseteq \tau_F(E')$ or $|\tau_F(E)| \leq |\tau_F(E')|$ is also possible by a deterministic polynomial-time algorithm. □

Proposition 110. r-σ EXTENSION RANKING *and* r-c-σ EXTENSION RANKING *with* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$ *are in* **P**.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. To compare E and E' with respect to $\tau \in \{\text{r-}\sigma, \text{r-c-}\sigma\}$ with $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{co\text{-}pr}, \mathsf{sst}\}$, we compare these two sets at most four times with respect to the base relations and this is possible in **P**, as shown in Proposition 109. □

Proposition 111. τ EXTENSION RANKING *for* $\tau \in \{\sqsupseteq^{\mathsf{nonatt}}, \sqsupseteq^{\mathsf{strdef}}\}$ *is in* **P**.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$.

"$\sqsupseteq^{\mathsf{nonatt}}$": First, we argue that E_F^- is computable by a deterministic polynomial-time algorithm. In the proof Proof for Proposition 109, we have shown that E_F^+ is computable by a deterministic polynomial-time algorithm. We use a similar approach for E_F^-. We need the adjacency matrix for F and then we can compute a_F^- for argument $a \in E$. We take the row of a and add all arguments with a value of 1. For E_F^-, take the union of all the elements. So, we calculate $|E \setminus (E_F^- \cap E')|$ by a deterministic polynomial-time algorithm, therefore $\sqsupseteq^{\mathsf{nonatt}}$ EXTENSION RANKING is in **P**.

"$\sqsupseteq^{\mathsf{strdef}}$" We restrict F to $F' = (\{E \cup E'\}, \{R \cap (E \cup E')\})$. Note that $sd_F(E, E') = sd_{F'}(E, E')$. Assume the contrary, assume there is an argument $a \in A$ such that $a \in sd_F(E, E')$ and $a \notin sd_{F'}(E, E')$. Meaning we have removed the strong defence of a in F'. So there has to be an attack $(c, b) \in R$ with $c \in E$ and $b \in E'$ with $(b, a) \in R'$, but $(c, b) \notin R'$. However, this attack would not be removed in F', hence a is still strongly defended by E from b in F'. Next assume that there is an argument $a' \in A$ such that $a' \notin sd_F(E, E')$ and $a' \in sd_{F'}(E, E')$. So we add the strong defence to a. However this is impossible since we only remove attacks and therefore we cannot add a new strong defence. Thus, $sd_F(E, E') = sd_{F'}(E, E')$.

We solve the credulous acceptance problems for these arguments. Caminada and Dunne (2019) have shown that credulous acceptance for strong admissibility is in **P**. We use the algorithm of Caminada and Dunne (2019) at most $|A|$ times and then compare the sums of E and E'. So, $\sqsupseteq^{\mathsf{strdef}}$ EXTENSION RANKING is in **P**. □

Proposition 112. *Let $F = (A, R)$ be an AF, $E, E' \subseteq A$, and $(\sqsupseteq^{\tau_1}_F, \ldots, \sqsupseteq^{\tau_n}_F)$ be a sequence of base relations, then:*

$$E \sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E' \text{ if and only if } E \sqsupseteq_F^{\mathsf{cope}^+(\tau_1,\ldots,\tau_n)} E'$$

Proof. Let $F = (A, R)$ be an AF, $E, E' \subseteq A$, and $(\sqsupseteq^{\tau_1}_F, \ldots, \sqsupseteq^{\tau_n}_F)$ be a sequence of base relations. Brandt et al. (2016b) have argued that there is a positive affine transformation function such that $Copeland(x) = \alpha Copeland^+(x) + \beta$, where $\alpha > 0$ and β are real number constants. Thus, if $E \sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E'$ then $E \sqsupseteq_F^{\mathsf{cope}^+(\tau_1,\ldots,\tau_n)} E'$ and if $E \sqsupseteq_F^{\mathsf{cope}^+(\tau_1,\ldots,\tau_n)} E'$ then $E \sqsupseteq_F^{\mathsf{cope}(\tau_1,\ldots,\tau_n)} E'$. □

Proposition 113. *If $\succsim$ RANKING is decidable in $\mathcal{C}$, then #COPELAND$^{\text{WIN}}_{\succsim}$ and #COPELAND$^{\text{DRAW}}_{\succsim}$ are in $\# \cdot \mathcal{C}$.*

Proof. Let $\mathcal{A}$ be a set of alternatives and $x \in \mathcal{A}$.

#COPELAND$^{\text{WIN}}_{\succsim}$: For each $y \in \mathcal{A}$ such that $x \succsim y$ and $y \not\succsim x$, we count up by one.

#COPELAND$^{\text{DRAW}}_{\succsim}$: For every pair (y_i, y_j) such that $x \succsim y_i$, $x \succsim y_j$, $y_i \succsim x$, and $y_j \succsim x$, we count up by one.

If we can verify $x \succsim y$ in $\mathcal{C}$, then #COPELAND$^{\text{WIN}}_{\succsim}$ and #COPELAND$^{\text{DRAW}}_{\succsim}$ are in $\# \cdot \mathcal{C}$. □

Proposition 114. *If τ EXTENSION RANKING for $\tau \in \{\tau_1, \ldots, \tau_n\}$ is in $\mathcal{C}$, then* $\mathsf{cope}(\sqsupseteq^{\tau_1}, \ldots, \sqsupseteq^{\tau_n})$ EXTENSION RANKING *is in* $\mathbf{P}^{\# \cdot \mathcal{C}}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. To decide $E \sqsupseteq_F^{\mathsf{cope} \sqsupseteq^{\tau_1},\ldots,\sqsupseteq^{\tau_n}} E'$, we use the alternative definition of the Copeland Score. Thus, we compute $Copeland^+(E)_{\sqsupseteq^{\tau_i}}$ and $Copeland^+(E')_{\sqsupseteq^{\tau_i}}$ for each $\tau_i \in \{\tau_1, \ldots, \tau_n\}$. For that, we need to count the number of times E and E' are strictly more plausible to accept than other sets and the number of times E and E' are equally plausible to accept with respect to other sets. These counts are possible in $\# \cdot \mathcal{C}$ if $E \sqsupseteq_F^{\tau_i} E'$ is decidable in $\mathcal{C}$. Hence, we access $\# \cdot \mathcal{C}$ oracles $4n$ times.

If $\sum_{i=1}^n Copeland^+(E)_{\tau_i} \geq \sum_{i=1}^n Copeland^+(E')_{\tau_i}$ then $E \sqsupseteq_F^{\mathsf{cope}^+(\sqsupseteq^{\tau_1},\ldots,\sqsupseteq^{\tau_n})} E'$. Because of Proposition 112:

$$E \sqsupseteq_F^{\mathsf{cope}(\sqsupseteq^{\tau_1},\ldots,\sqsupseteq^{\tau_n})} E'$$

Hence, we return `Yes` otherwise we return `No`. Thus, if deciding $E \sqsupseteq_F^{\tau_i} E'$ is in $\mathcal{C}$ for $\tau_i \in \{\tau_1, \dots, \tau_n\}$, then we can decide $E \sqsupseteq_F^{\mathsf{cope}(\sqsupseteq^{\tau_1},\dots,\sqsupseteq^{\tau_n})} E'$ in $\mathbf{P}^{\#\cdot\mathcal{C}}$. This decision is equivalent to the decision problem τ_i EXTENSION RANKING. □

Lemma 7. TOLERATION *is* **NP**-*complete.*

Proof. Let $F = (A, R)$ be an AF, $a, b \in A$ and $(a, b) \in R$ an attack. In order to show that (a, b) is tolerated by R, we need to find a set $E \subseteq A$ such that:

1. (a, b) is verified by E
2. Every $r' \in R$ is satisfied by E
3. For every argument $c \in A$ attack admissibility is satisfied.

In order for 1. to hold, we have to have: $a \in E$ and $b \notin E$ and for 2. it has to hold that $E \in \mathsf{cf}(F)$. Finally, for 3. that for every $c \in A$, either $c \in E$ or there exists an argument $d \in c_F^-$ such that $d \in E$. So, E has to be conflict-free and for every argument c, E either contains c or contains an attacker of c, thus E has to be a stable extension in F. Since $a \in E$, we know that we are searching for a stable extension that contains argument a. Thus, to tolerate (a, b) with R we have to decide if a is credulous accepted with respect to stable semantics in F. Since CRED$_\sigma$ is **NP**-complete (Dvorák and Dunne, 2017), TOLERATION is also **NP**-complete. □

Lemma 8. *Let $F = (A, R)$ be an AF and $(R_0, \dots, R_n)$ the Z-attack-Partitioning of F. For $0 \leq i \leq n$ $R_i = R_0'$ for Z-attack-Partitioning $(R_0', \dots, R_m')$ of $F' = (A, R \setminus \{R_0 \cup \dots \cup R_{i-1}\})$.*

Proof. Let $F = (A, R)$ be an AF and $(R_0, \dots, R_n)$ the Z-attack-Partitioning of F. Consider R_i and $F' = (A, R \setminus \{R_0 \cup \dots \cup R_{i-1}\})$, then by definition of Z-attack-Partitioning, $(R_1, \dots, R_n)$ is the Z-attack-Partitioning of $R \setminus (R_0)$. We can extend this to: $(R_i, \dots, R_n)$ is the Z-attack-Partitioning of $R \setminus (R_0 \cup \dots R_{i-1})$. This means that every attack $r \in R_i$ is tolerated by $R \setminus (R_0 \cup \dots R_{i-1})$. Every attack $r \in R_0'$ is tolerated by $R \setminus (R_0 \cup \dots R_{i-1})$ and this entails $R_0' = R_i$. □

Proposition 115. $\sqsupseteq^{\kappa^Z}$ EXTENSION RANKING *is in* Δ_2^p.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$. To decide $E \sqsupseteq_F^{\kappa^Z} E'$, we compute $\kappa_F^Z(E)$ and $\kappa_F^Z(E')$. To compute these two values, we compute the violated attacks by E and E'. This can be done by computing the conflicts of these sets, i. e. $\mathsf{Conflicts}_F(E)$ and $\mathsf{Conflicts}_F(E')$. Then for every attack $r \in \mathsf{Conflicts}_F(E)$ and $r' \in \mathsf{Conflicts}_F(E')$, we compute $Z_R(r)$ and $Z_R(r')$, respectively. We can do this by asking if r (r') is tolerated by R_i. Lemma 7 shows that this problem is **NP**-complete. For every argument, we ask at most $|R|$ times, since a Z-attack-Partitioning of F can have at most $|R|$ layers. Thus, we access an **NP**-oracle at most $2 * |R|^2$ times. Showing that $\sqsupseteq^{\kappa^Z}$ EXTENSION RANKING is in $\mathbf{P}^{\mathbf{NP}} = \Delta_2^p$. □

Proposition 116. *Comparison relations* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$ *are computable by a deterministic polynomial-time algorithm.*

Proof. Let $v = \{v_1, \dots, v_n\}$ and $v' = \{v'_1, \dots, v'_m\}$ be vectors of numbers.

"$\Box = \mathsf{sum}$": To calculate $\mathsf{sum}(v)$, we to sum up all elements of v, $\mathsf{sum}(v) = \sum_{i=1}^{n} v_i$. Then we compare two numbers. This is possible by a deterministic polynomial-time algorithm.

"$\Box = \mathsf{max}$": To find the maximal element, we check the vector once. So, we can solve this by a deterministic polynomial-time algorithm.

"$\Box = \mathsf{min}$": To find the minimal element, we check the vector once. So, we can solve this by a deterministic polynomial-time algorithm.

"$\Box = \mathsf{leximax}$": To compute $\mathsf{leximax}(v)$, we sort the vector in descending order. This is possible in $\mathcal{O}(n\ log\ n)$. Then we need at most n comparisons. So, this is possible by a deterministic polynomial-time algorithm.

"$\Box = \mathsf{leximin}$": To compute $\mathsf{leximax}(v)$, we sort the vector in ascending order. This is possible in $\mathcal{O}(n\ log\ n)$. Then we need at most n comparisons. So, this is possible by a deterministic polynomial-time algorithm. □

Proposition 117. *If the argument-ranking semantics* ρ *is computable in complexity class* $\mathcal{C}$*, then* $\mathsf{GCB}_{\rho,\Box}$ EXTENSION RANKING *is decidable in* $\mathbf{P}^{\mathcal{C}}$ *for* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.

Proof. Let $F = (A, R)$ be an AF, ρ an argument-ranking semantics, and $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. Assume two sets $E, E' \subseteq A$. To decide $E \sqsupseteq_F^{\mathsf{GCB}_{\rho,\Box}} E'$, we compute $\Box(\rho_F(E)) \geq^{\Box} \Box(\rho_F(E'))$. Proposition 116 shows that $\Box(E) \geq^{\Box} \Box(E')$, is computable in **P**. To calculate $\rho_F(E)$ and $\rho_F(E')$, we compute the argument ranking ρ_F. For that purpose, we decide ρ ARGUMENT RANKING for every pair of arguments. Thus, we utilise a $\mathcal{C}$ oracle at most $|A|^2$ times. Implying, that $\mathsf{GCB}_{\rho,\Box}$ EXTENSION RANKING is decidable in $\mathbf{P}^{\mathcal{C}}$ for $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. □

Proposition 118. $\mathsf{GCB}_{ne_\sigma,\Box}$ EXTENSION RANKING *for* $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{st}\}$ *is in* $\mathbf{P}^{\#\cdot\mathbf{P}}$ *and for* $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$ *in* $\mathbf{P}^{\#\cdot\mathbf{coNP}}$ *for* $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$.

Proof. Let $F = (A, R)$ be an AF, ne_σ the gradual semantics for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{pr}, \mathsf{st}, \mathsf{sst}\}$, and $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. For ne_σ, we compute $|\{E \in \sigma(F) \mid a \in E\}|$ for every $a \in A$ and this is the output of the counting problem $\#\textsc{Cred}_\sigma$ for AF F and argument a. This counting problem is in $\# \cdot \mathbf{P}$ for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{st}\}$ and in $\# \cdot \mathbf{coNP}$ for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$. Proposition 117 gives us then the complexity results of $\mathsf{GCB}_{ne_\sigma,\Box}$ EXTENSION RANKING for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{st}\}$. This problem is in $\mathbf{P}^{\#\cdot\mathbf{P}}$ and for $\sigma \in \{\mathsf{pr}, \mathsf{sst}\}$ in $\mathbf{P}^{\#\cdot\mathbf{coNP}}$ for $\Box \in \{\mathsf{sum}, \mathsf{max}, \mathsf{min}, \mathsf{leximax}, \mathsf{leximin}\}$. □

Proposition 119. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* Hoare_ρ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$.
To decide Hoare_ρ EXTENSION RANKING, we present a deterministic polynomial-time algorithm. This algorithm utilises an argument-ranking semantics ρ as an oracle function.

The algorithm begins by iterating over each argument in E'. For each argument $b \in E'$, it checks whether there exists an argument $a \in E$ such that a is stronger than b with respect to the argument-ranking semantics ρ. If the oracle returns `Yes` for $a \succeq^\rho_F b$, the algorithm proceeds to the next argument in E'. If no stronger argument a is found for any b, the algorithm terminates and returns `No`.

The algorithm consists of a nested for-loop, iterating over each element in E and E'. Given that $E, E' \subseteq A$, the algorithm's runtime is $\mathcal{O}(|A|^2)$. Therefore, Hoare_ρ EXTENSION RANKING is in $\mathbf{P}^{\mathcal{C}}$ if oracle ρ runs in $\mathcal{C}$. □

Proposition 120. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* Kelly_ρ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$.
To decide Kelly_ρ EXTENSION RANKING, we describe a deterministic polynomial-time algorithm.This algorithm utilises an argument-ranking semantics ρ as an oracle function.

We iterate over each argument of E and E'. If we find an argument of E' that is stronger than an argument of E, we stop our search and return `No`. If we never find such an argument in E', we return `Yes`.

The algorithm consists out of a nested for-loop that iterates over each argument of E and E'. It has a runtime of $\mathcal{O}(|A|^2|)$ and Kelly_ρ EXTENSION RANKING is in $\mathbf{P}^{\mathcal{C}}$ if the oracle ρ runs in $\mathcal{C}$. □

Proposition 121. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* $\mathsf{Fishburn}_\rho$ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$.
To decide $\mathsf{Fishburn}_\rho$ EXTENSION RANKING, we present a deterministic polynomial-time algorithm. Assume an argument-ranking semantics ρ as an oracle function.

The algorithm starts with an iteration over all arguments that are only in E ($E \setminus E'$) and then iterates over all arguments that are only in E'. A final loop to iterate over the argument which are in both sets. If we find an argument in $E' \setminus E$ that is stronger than any argument in $E \setminus E'$ or $E \cap E'$, then E is not at least as plausible to accept as E', so we return `No`. If we find an argument in $E \cap E'$ that is stronger than an argument in $E \setminus E'$, then we return `No`. So, the algorithm consists of three nested for-loops for which the maximum number of iterations is $|A|$. Thus, the algorithm runs in $\mathcal{O}(|A|^3)$ and therefore $\mathsf{Fishburn}_\rho$ EXTENSION RANKING is in $\mathbf{P}^{\mathcal{C}}$ if the oracle ρ runs in $\mathcal{C}$. □

Proposition 122. *If the argument-ranking semantics ρ is computable in complexity class $\mathcal{C}$, then* Eli_ρ EXTENSION RANKING *is in* $\mathbf{P}^{\mathcal{C}}$.

Proof. Let $F = (A, R)$ be an AF and $E, E' \subseteq A$.
To decide Eli_ρ EXTENSION RANKING, we present a deterministic polynomial-time algorithm. This algorithm utilises an argument-ranking semantics ρ as an oracle function.

We start by iterating over all arguments in E'. We try to find an argument in E' that is weaker than any argument from E. If we find such an argument, we return `Yes`, otherwise we return `No`. The algorithm consists of one nested for-loop and therefore runs in $\mathcal{O}(|A|^2)$. So Eli_ρ EXTENSION RANKING is in $\mathbf{P}^{\mathcal{C}}$ if the oracle ρ runs in $\mathcal{C}$. $\square$

ADDITIONAL INFORMATION B

Times $X \cup \{a\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{b\} : 28$	Times $X \cup \{a\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{c\} : 16$
Times $X \cup \{a\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{d\} : 14$	Times $X \cup \{a\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{e\} : 16$
Times $X \cup \{a\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{f\} : 16$	Times $X \cup \{a\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{g\} : 14$
Times $X \cup \{b\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{a\} : 0$	Times $X \cup \{b\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{c\} : 14$
Times $X \cup \{b\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{d\} : 4$	Times $X \cup \{b\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{e\} : 8$
Times $X \cup \{b\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{f\} : 6$	Times $X \cup \{b\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{g\} : 4$
Times $X \cup \{c\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{a\} : 0$	Times $X \cup \{c\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{b\} : 2$
Times $X \cup \{c\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{d\} : 0$	Times $X \cup \{c\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{e\} : 4$
Times $X \cup \{c\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{f\} : 3$	Times $X \cup \{c\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{g\} : 0$
Times $X \cup \{d\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{a\} : 6$	Times $X \cup \{d\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{b\} : 8$
Times $X \cup \{d\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{c\} : 14$	Times $X \cup \{d\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{e\} : 8$
Times $X \cup \{d\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{f\} : 16$	Times $X \cup \{d\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{g\} : 2$
Times $X \cup \{e\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{a\} : 0$	Times $X \cup \{e\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{b\} : 0$
Times $X \cup \{e\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{c\} : 0$	Times $X \cup \{e\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{d\} : 0$
Times $X \cup \{e\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{f\} : 0$	Times $X \cup \{e\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{g\} : 0$
Times $X \cup \{f\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{a\} : 2$	Times $X \cup \{f\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{b\} : 2$
Times $X \cup \{f\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{c\} : 6$	Times $X \cup \{f\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{d\} : 0$
Times $X \cup \{f\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{e\} : 4$	Times $X \cup \{f\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{g\} : 0$
Times $X \cup \{g\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{a\} : 8$	Times $X \cup \{g\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{b\} : 15$
Times $X \cup \{g\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{c\} : 16$	Times $X \cup \{g\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{d\} : 10$
Times $X \cup \{g\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{e\} : 16$	Times $X \cup \{g\} \sqsupset^{\text{r-co}}_{F_4} X \cup \{f\} : 32$

Table B.1: CPM Scores for AF F_4 for Example 148, where a line can be interpreted as, how many set X does there exists such that the condition holds?

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NOTATIONS

$\mathcal{R}$	General preorder
$\mathcal{P}(X)$	Powerset of X
AF	Abstract argumentation framework
A	Set of arguments of an AF
R	Set of attacks of an AF
a_F^-	Set of attackers of argument a in AF F
a_F^+	Set of arguments attacked by argument a in AF F
$\mathcal{F}_F(E)$	Characteristic function aplied to set E wrt. AF F
$\mathsf{cf}(F)$	Set of conflict-free sets of AF F
$\mathsf{ad}(F)$	Set of admissible sets of AF F
$\mathsf{co}(F)$	Set of complete extensions of AF F
$\mathsf{pr}(F)$	Set of preferred extensions of AF F
$\mathsf{gr}(F)$	Set of grounded extensions of AF F
$\mathsf{st}(F)$	Set of stable extensions of AF F
$\mathsf{sst}(F)$	Set of semi-stable extensions of AF F
$\gamma(F)$	Isomorphism of F
σ	generals extension semantics
$\mathfrak{F}_F(E)$	Characteristic functions for AFs applied to set E wrt. AF F
$\mathsf{cred}_\sigma(F)$	Credulously accepted arguments of F wrt. σ
$\mathsf{skep}_\sigma(F)$	Skeptically accepted arguments of F wrt. σ
$\mathsf{rej}_\sigma(F)$	Rejected arguments of F wrt. σ
ρ	general argument-ranking semantics
Bbs	Burden-based argument-ranking semantics
Cat	h-categoriser argument-ranking semantics
Ser	serialisability argument-ranking semantics
$\mathsf{IS}(F)$	Initial sets of AF F

$cc(F)$	connected components of AF F
Abs	Abstraction
In	Independence
VP	Void Precedence
SC	Self-Contradiction
CP	Cardinality Precedence
QP	Quality Precedence
CT	Counter-Transitivity
SCT	Strict Counter-Transitivity
DP	Defense Precedence
DDP	Distributed Defense Precedence
NaE	Non-attacked Equivalence
AvsFD	Attack vs. Full Defense
σ*-C*	σ-Compatibility
σ*-sk-C*	σ-sk-Compatibility
wσ*-S*	weak σ-Support
sσ*-S*	strong σ-Support
UR	Universally Realisable
τ	general extension-ranking semantics
LD^σ	least-discriminating extension-ranking semantics wrt. σ
$\mathtt{max}_\sigma(F)$	Most plausible sets wrt. σ in AF F
σ*-soundness*	Principle σ*-soundness* wrt. extension semantics σ
σ*-completeness*	Principle σ*-completeness* wrt. extension semantics σ
σ*-generalisation*	Principle σ*-generalisation* wrt. extension semantics σ
respect conflicts	Principle *respect conflicts*
Conflicts	Base relation conflicts
Undef	Base relation undefended arguments
Condef	Base relation consistently defended and not in
Unatt	Base relation unattacked arguments
$\mathfrak{F}^*_F(E)$	Modified characteristic function applied to set E wrt. AF F
c-τ	cardinality-based version of base relation τ for $\tau \in \{\mathsf{Conflicts}, \mathsf{Undef}, \mathsf{Condef}, \mathsf{Unatt}\}$
$sd_F(a, E, E')$	Argument a is strongly defended by set E from set E' in AF F
lex	lexicographic combination
r-ad	Admissible extension-ranking semantics
r-co	Complete extension-ranking semantics
r-gr	Grounded extension-ranking semantics

r-pr	Preferred extension-ranking semantics
r-sst	Semi-Stable extension-ranking semantics
r-c-σ	Cardinality-based extension-ranking semantics for $\sigma \in \{\mathsf{ad}, \mathsf{co}, \mathsf{gr}, \mathsf{pr}, \mathsf{sst}\}$
mini	Minimality base relation
maxi	Maximality base relation
$\mathcal{A}$	Alternatives in an election
$\mathcal{V}$	Set of voters in an election
$\mathcal{VR}$	General voting rule
cope	Copeland-based combination
OCF	General ordinal conditional function
ι	General lifting operator
$\mathsf{Aqc}_{\rho,F}(E, E')$	Argument quality count between sets E and E' wrt. ρ in F.
$\sqsupseteq_F^{\mathsf{Aqc}_\rho}$	Base relation argument quality count wrt. ρ in F.
Hoare_ρ	Hoare's extension-ranking semantics wrt. ρ.
$\varepsilon_F(a)$	general numerical evaluation function for argument a in AF F.
$ne_{\sigma,F}(a)$	Number of σ-extension argument a is part of in AF F.
$\square$	General comparison relation
$\mathsf{sum}(v)$	Summation of elements of sequence v
$\mathsf{max}(v)$	Maximal element of sequence v
$\mathsf{min}(v)$	Minimal element of sequence v
$\mathsf{leximax}(v)$	Comparison relation lexicographical maxima
$\mathsf{leximin}(v)$	Comparison relation lexicographical minima
$\mathsf{GCB}_{\rho,\square}$	General comparison-based extension-ranking semantics based on argument-ranking semantics ρ and comparison relation $\square$.
$sv_F^\rho(a)$	Numerical strength value of argument a in AF F wrt. argument-ranking semantics ρ.
Kelly_ρ	Kelly's extension-ranking semantics based on argument-ranking semantics ρ.
$\mathsf{Fishburn}_\rho$	Fishburn's extension-ranking semantics based on argument-ranking semantics ρ.
Eli_ρ	Elitist extension-ranking semantics based on argument-ranking semantics ρ.
ξ	general social ranking function
$\mathsf{ST}_\mathcal{R}$	singleton social ranking function with respect to $\mathcal{R}$
$\mathsf{rank}_\mathcal{R}(X)$	Rank of X in preorder $\mathcal{R}$
$\mathsf{lex\text{-}cel}_\mathcal{R}$	lexicographic excellence operator with respect to $\mathcal{R}$

$\mathsf{CPM}_{\mathcal{R}}$	Ceteris Paribus Majority Solution with respect to $\mathcal{R}$
$U_x(S)$	set of subsets of 2^S which do not contain x
$m_x^X(\mathcal{R})$	ordinal marginal contribution of x to X with respect to $\mathcal{R}$
$u_x^{+,\mathcal{R}}(S)$	set of $X \in U_x(S)$ such that $m_x^X(\mathcal{R}) = 1$
$u_x^{-,\mathcal{R}}(S)$	set of $X \in U_x(S)$ such that $m_x^X(\mathcal{R}) = -1$
$\mathsf{Banz}_{\mathcal{R}}$	ordinal Banzhaf solution with respect to $\mathcal{R}$
$\mathcal{C}$	complexity class
P	complexity class of decision problem solvable by a *deterministic polynomial-time algorithm*
NP	complexity class of decision problem solvable in an *non-deterministic polynomial-time algortihm*
τ EXTENSION RANKING	extension ranking problem for extension-ranking semantics τ