

Axiomatics of Restricted Choices by Linear Orders of Sets with Minimum as Fallback

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Abstract. We study how linear orders can be employed to realise choice functions for which the set of potential choices is restricted, i.e., the possible choice is not possible among the full powerset of all alternatives. In such restricted settings, constructing a choice function via a relation on the alternatives is not always possible. However, we show that one can always construct a choice function via a linear order on sets of alternatives, even when a fallback value is encoded as the minimal element in the linear order. The axiomatics of such choice functions are presented for the general case and the case of union-closed input restrictions. Restricted choice structures have applications in knowledge representation and reasoning, and here we discuss their applications for theory change and abstract argumentation.

Keywords: choice functions · theory change · argumentation

1 Introduction

A (classical) choice structure $\langle A, \mathbb{S} \rangle$ consists of a set of alternatives A and a set of subsets $\mathbb{S} \subseteq \mathcal{P}(A)$ of A . A (classical) choice function $O : \mathbb{S} \rightarrow \mathcal{P}(A)$ for $\langle A, \mathbb{S} \rangle$ maps each set $S \in \mathbb{S}$ to a subset of S . Some authors also demand that $O(S)$ is non-empty if and only if S is non-empty and in some communities, choice functions are required to output a singleton whenever S is non-empty.

Example 1.1. We are planning to buy snacks in a supermarket for the evening. From experience, one knows that the typical snacks that are available in a supermarket are $A = \{\text{chocolate}, \text{nachos}, \text{pretzels}, \text{dips}, \text{chillies}\}$. We determine that we want to buy only salty snacks $S = \{\text{nachos}, \text{pretzels}, \text{dips}\}$. Because one does not want to buy all available options, one might decide to choose from one of the available options $O(S) = \{\text{pretzels}, \text{nachos}\}$. ■

Work on choices has applications in, e.g., social choice theory and economics [33], mathematics and logic [37], and knowledge representation (KR) [27,18]. Here, we are aiming at its impact on KR, where approaches to choice are connected to nonmonotonic reasoning, belief change, update, and conditionals [26]. Research in semantics of, e.g., belief revision, shows that revisions satisfy desired properties exactly when the result of revision is chosen by employing an underlying

preference structure [20]. However, as observed, the connection between known axiomatizations and choice developed, which hold, e.g., for propositional logic, does not carry over to certain settings. Examples are preferential team-based logics, where the axiomatization turns out to be difficult [30], Horn logic [9,10], or belief change in epistemic spaces, where belief change operators are not realizable in certain circumstances [32,28]. The rationale why classical connections do not carry over to these settings is that the output of potential choices is restricted in these settings. The choice function cannot simply output an element of the full powerset over the alternatives, i.e., $O(S)$ “should” be a certain set, but $O(S)$ is not expressible, e.g. in Horn logic. The following Example 1.2 illustrates that one is often more restricted in choice on the output side than the original setting of choice permits, i.e., the codomain of a choice $O(S)$ is not the full powerset of all alternatives.

Example 1.2 (Continued from Example 1.1). Originally, in Example 1.1, we decided to buy **pretzels** and **nachos**, but nothing else. However, the supermarket we are visiting offers **nachos**, **dips** and **chillies** only in a bundle together. This means that the choice $O(S) = \{\text{pretzels}, \text{nachos}\}$ given in Example 1.1 is not valid in this setting, because the supermarket has no such offer in their stock. ■

Existence and non-existence of choice functions in the classical setting of choice has been studied extensively [8]. The existence of any choice functions is not guaranteed when considering arbitrary sets within the Zermelo–Fraenkel (ZF) set theory [19]. This can be resolved by adding the Axiom of Choice (ZFC), which enforces the existence of choice functions (that output only singletons):

(Axiom of Choice) For each set of non-empty sets $\mathbb{S}$, there exists a function $f : \mathbb{S} \rightarrow \bigcup_{S \in \mathbb{S}} S$ with $f(S) \in S$ for every $S \in \mathbb{S}$.

From social choice theory [21], it is well-known that already mild assumptions about the representation of choice functions lead to the non-existence of choice functions, even when one assumes ZFC. An often considered case is the construction of choice functions via a relation. This is done by considering a preference relation $\leq \subseteq A \times A$ on the alternatives A . A choice function O is then defined by letting $O(S)$ be the most preferred elements within S according to $\leq$, which we write as $O(S) = \min(S, \leq)$.⁴ Many authors follow the suggestion from Sen’s seminal work [33] to consider unrestricted inputs, i.e., having $\mathbb{S} = \mathcal{P}(A)$. Effectively, many theoretical results for choice functions are given under this condition.

In this paper, we study in a general manner the setting where choices are limited to a set of choices $\mathbb{E} \subseteq \mathcal{P}(A)$ (which we call the realizable choices) and where the sets to choose from are limited to $\mathbb{S} \subseteq \mathcal{P}(A)$. A *restricted choice function* is a function $C : \mathbb{S} \rightarrow \mathbb{E}$ where $C(S) \subseteq S$ holds, whenever possible. Formally, the setting of restricted choice functions enables us to model the situation from Example 1.2 easily.

⁴ We follow here the convention that being smaller in $\leq$ corresponds to being more preferred, which is in line with the typical reading of orders in belief revision and nonmonotonic reasoning. Technically, one could also consider everything from a dual perspective, where the larger elements are the more preferred elements.

Example 1.3 (Continued from Example 1.2). We are only permitting choices where **nachos**, **dips** and **chillies** appear together, i.e., by setting $\mathbb{E}$ to

$$\begin{aligned} \mathbb{E} &= \{X \subseteq A \mid (\{\mathbf{nachos}, \mathbf{dips}, \mathbf{chillies}\} \cap X \neq \emptyset) \\ &\quad \text{implies } \{\mathbf{nachos}, \mathbf{dips}, \mathbf{chillies}\} \subseteq X\} \\ &= \{\{\mathbf{pretzels}\}, \{\mathbf{chocolate}\}, \{\mathbf{pretzels}, \mathbf{chocolate}\}, \\ &\quad \{\mathbf{nachos}, \mathbf{dips}, \mathbf{chillies}\}, \{\mathbf{pretzels}, \mathbf{nachos}, \mathbf{dips}, \mathbf{chillies}\}, \\ &\quad \{\mathbf{chocolate}, \mathbf{nachos}, \mathbf{dips}, \mathbf{chillies}\}, A\} . \end{aligned}$$

Consequently, the only valid output $C(S)$ for $S = \{\mathbf{pretzels}, \mathbf{nachos}, \mathbf{dips}\}$ from Example 1.1 is $C(S) = \{\mathbf{pretzels}\}$ because we decided beforehand not to buy **chillies**, i.e., $\mathbf{chillies} \notin S$. Moreover, we have thought carefully about snacks and come to the conclusion that we only want **nachos** when we can also get **dips**. One can model such personal restrictions on the input side, e.g., by considering restricted choice functions for $\mathbb{S} = \{S \subseteq X \mid \mathbf{nachos} \in S \Rightarrow \mathbf{dips} \in S\}$. ■

The setting of restricted choice is a powerful generalization of the classical setting that permits specification of restrictions on the output side. Two basic conceptual problems arise, which we resolve in this paper:

- Unrealisable choices
- Existence and representation of restricted choice functions

The latter is because the axiom of choice does not immediately guarantee the existence of a restricted choice function. Specifically, it is not clear how a choice function for restricted choice structures can be represented or constructed via orders. Here, we show that an efficient way is to employ a linear order on sets of alternatives. The problem of unrealisable choices arises, because there are situations where it is impossible to yield a choice for a set $S \in \mathbb{S}$ because there is no $E \in \mathbb{E}$ in the co-domain of C such that $E \subseteq S$. However, as C is a function, there has to be an output for $C(S)$.

Example 1.4. Consider again the situation discussed in Example 1.3 where **nachos** are bundled with **dips** and **chillies**. For example, for $S' = \{\mathbf{nachos}, \mathbf{dips}\}$ there is no $E \in \mathbb{E}$ such that $E \subseteq S'$.

Here, we solve the problem of unrealisable choices by output a fallback value. For that, we encode the fallback value as the minimal element in a linear order.

We demonstrate how restricted choice functions can be employed in theory change and abstract argumentation. Specifically, we show that our representation theorem for linear choice functions carries over to theory change and abstract argumentation. We see that linear choice functions lead to a natural general approach to theory change operators. In abstract argumentation, we show that employing choice functions leads to a new approach to argumentation semantics, which has not been explored and generalizes extension selection [22,7,34].

In summary, the main content of the paper is the following, which we also consider as the main contributions of this paper⁵:

- [Restricted Choice Structures and their Choice Functions] We introduce the concepts of restricted choice structures and (restricted) choice functions for restricted choice structures. Restricted choice structures provide a more expressive setting than the original choice setting that permits taking unrealisable choices into account.
- [Linear Choice Functions] We present a uniform way of constructing restricted choice functions by employing a linear order on the available outcomes. We show that this construction provides a restricted choice function for any restricted choice structure, a feature that typical construction methods via orders on alternatives do not guarantee. Moreover, the full axiomatization of linear order-based choice functions is provided for both union-closed restricted choice structures and arbitrary restricted choice structures.
- [Applications in KR] We discuss the application of restricted choice structures and their choice functions in knowledge representation and reasoning (KR). We discuss the application in theory revision and argumentation.

In the next section, we start by presenting the background of order theory. In Section 3 we formally introduce restricted choice structures and their choice functions. Linear choice functions are introduced in Section 4. The axiomatics of linear choice functions is given in Section 5. Section 6 is dedicated to explore applications of the restricted choice structures and linear choice functions in knowledge representation and reasoning; namely theory change (Section 6.1) and argumentation (Section 6.2). The conclusion of this paper is given in Section 7.

2 Background on Relations and Order Theory

We use $\mathbb{N}$ for the natural numbers including 0, and $\mathbb{N}^+$ for the natural numbers excluding 0. The powerset of a set X is denoted by $\mathcal{P}(X)$. In the following, we present the background on basic notions of relations and order theory. Moreover, we present some basic background in extensions of relations to orders.

Relations and Orders. A (binary) relation on a set X is a subset of $X \times X$. We will often use order symbols for relations and write them infix, e.g., $x \preceq y$ means the same as $(x, y) \in \preceq$ for a relation $\preceq$. With $\prec$ we denote the strict part of a relation $\preceq$ on X and with $\simeq$ we denote the equivalent part of $\preceq$, i.e., $\simeq = \preceq \cap \{(x_2, x_1) \in X \times X \mid x_1 \preceq x_2\}$ and $\prec = \preceq \setminus \simeq$. The following properties of a relation $\preceq \subseteq X \times X$ are considered in this article:

- (reflexive) $x \preceq x$
- (total)⁶ $x_1 \preceq x_2$
- (antisymmetric) $x_1 \preceq x_2$ and $x_2 \preceq x_1$ imply $x_1 = x_2$
- (transitive) $x_1 \preceq x_2$ and $x_2 \preceq x_3$ imply $x_1 \preceq x_3$
- (consistent) $x_0 \preceq x_1, \dots, x_{n-1} \preceq x_n$ implies $x_n \not\prec x_0$

⁵ The full proofs are given in the supplement of the arXiv version of this paper [31].

A *preorder* $\preceq$ on a set X is a relation $\preceq \subseteq X \times X$ such that $\preceq$ is reflexive and transitive. We also consider *total preorders*, i.e., preorders that also satisfy totality. A *linear order* on a set X is a total preorder on X that is additionally antisymmetric. The consistency property is due to Suzumura and has a central place in order theory, as it guarantees the existence of an order-extension [35]. A subtle aspect of the consistency property is that one demands $x_n \not\prec x_0$ instead of $x_n \not\preceq x_0$. This ensures that the consistency property does not apply when $x_n \simeq x_0$ holds. If $\preceq$ is transitive, then $\preceq$ is also consistent, and if $\preceq$ is total, then $\preceq$ is also reflexive.

Minimal Elements. We define two types of minimal elements for $\preceq \subseteq X \times X$, the (globally) minimal elements $\min(\preceq)$, and the minimal elements $\min(M, \preceq)$ with respect to a set M , which are given by:

$$\begin{aligned} \min(\preceq) &= \{ x \in X \mid x' \preceq x \text{ implies } x \preceq x' \text{ for all } x' \in X \} \\ \min(M, \preceq) &= \{ x \in M \cap X \mid x' \preceq x \text{ implies } x \preceq x' \text{ for all } x' \in M \} \end{aligned}$$

Note that for arbitrary relations (even on a finite set) there might be no minimal elements, i.e., $\min(\preceq)$ and $\min(M, \preceq)$ might be empty sets. However, if $\preceq$ is a total preorder on a finite (non-empty) set, then $\min(\preceq)$ is always non-empty, and $\min(M, \preceq)$ is only empty if M is empty. We say that $\preceq \subseteq X \times X$ is *well-founded* if for each $M \subseteq X$ holds $\min(M, \preceq) \neq \emptyset$.

We deal in this paper with relations on sets of sets $\mathcal{E} \subseteq \mathcal{P}(A)$ over some base-set A . For such a relation $\leq \subseteq \mathcal{E} \times \mathcal{E}$, we overload the notion of $\min(\cdot, \cdot)$ by extending it to elements from $\mathcal{P}(A)$. When $S \in \mathcal{P}(A)$ is an element from $\mathcal{P}(A)$, then we let $\min(S, \leq) = \min(\{E \in \mathcal{E} \mid E \subseteq S\}, \leq)$. That is, $\min(S, \leq)$ is the set of $\leq$ -minimal elements among all elements from $\mathcal{E}$ that are also subsets of S .

3 Restricted Choice Structures

We define restricted choice structures and their corresponding choice functions.

Definition 3.1. A (normal) *restricted choice structure* is a tuple $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$ where

- A is a set (the set of alternatives),
- $\mathbb{S} \subseteq \mathcal{P}(A)$ is a non-empty set of subsets of A (the domain), and
- $\mathbb{E} \subseteq \mathcal{P}(A)$ is a non-empty subset of $\mathbb{S}$ (the realizable choices) with $\mathbb{E} \subseteq \mathbb{S}$.

We call $\mathcal{R}$ (input) union-closed if $S_1 \cup S_2 \in \mathbb{S}$ holds for all $S_1, S_2 \in \mathbb{S}$.

Next, we define the notion of choice functions for restricted choice structures.

Definition 3.2. Let $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$ be a restricted choice structure. A function $C : \mathbb{S} \rightarrow \mathbb{E}$ is called a *choice function* for $\mathcal{R}$ if for each $S \in \mathbb{S}$ holds $C(S) = E \in \mathbb{E}$ with $E \subseteq S$, if such an $E \in \mathbb{E}$ exists. If $C(S) = K$ for all $S \in \mathbb{S}$ for which no $E \in \mathbb{E}$ with $E \subseteq S$ exists, we say that K is the *fallback (value) of C* .

Note that a (unrestricted) choice structure $\langle A, \mathbb{S} \rangle$ can **not** be simply reconstructed by taking $\langle A, \mathbb{S}, \mathcal{P}(\mathcal{P}(A)) \rangle$, as the latter violates $\mathbb{E} \subseteq \mathbb{S}$ and thus is not

⁶ This is also known as (strongly) *connected*.

a (normal) restricted choice structures. Instead, the restricted choice structure $\langle A, \mathbb{S}, \mathbb{E} \rangle$ with $\mathbb{E} = \bigcup_{S \in \mathbb{S}} \mathcal{P}(S)$ reconstructs $\langle A, \mathbb{S} \rangle$ in the following sense:

- (a) for every choice function $O : \mathbb{S} \rightarrow \mathcal{P}(A)$ for $\langle A, \mathbb{S} \rangle$, there is a choice function $C : \mathbb{S} \rightarrow \mathbb{E}$ for $\langle A, \mathbb{S}, \mathbb{E} \rangle$, such that $O(S) = C(S)$ for all $S \in \mathbb{S}$.
- (b) for every choice function $C : \mathbb{S} \rightarrow \mathbb{E}$ for $\langle A, \mathbb{S}, \mathbb{E} \rangle$, there is a choice function $O : \mathbb{S} \rightarrow \mathcal{P}(A)$ for $\langle A, \mathbb{S} \rangle$, such that $O(S) = C(S)$ for all $S \in \mathbb{S}$.

One might get the impression that the last condition of $\mathbb{E} \subseteq \mathbb{S}$ in Definition 3.1 is too restrictive, as one may imagine settings of restricted choice, where the condition $\mathbb{E} \subseteq \mathbb{S}$ does not hold. However, without $\mathbb{E} \subseteq \mathbb{S}$, we have to deal with (potentially) unwanted consequences for choice functions C . Examples are:

- $\mathbb{E}$ contains elements that are not in the image of any choice function at all;
- choice might be not chainable, e.g., $C(C(S))$ is undefined;
- $C(K)$ is undefined, while K is the fallback of C .

Because of that, we call a restricted choice structure without $\mathbb{E} \subseteq \mathbb{S}$ *non-normal*. In this paper, we refrain from considering non-normal restricted choice structures and assume normality for the remainder of the paper. However, non-normal restricted choice structures might be of interest in future work.

4 Linear Choice Functions

In this section, we show how to construct a choice function for a restricted choice structure by employing a linear order on the sets of realizable choice sets. Not all linear orders are suitable for the approach. The property we require is the existence of minima for all cases in which one wants to make choices. We call this property *smoothness*. Fallbacks will be encoded as the globally minimal element.

Definition 4.1 ($\mathcal{R}$ -smoothness, K -minimal). Let $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$ be a restricted choice structure, let $K \in \mathbb{E}$ and let $\leq \subseteq \mathcal{E} \times \mathcal{E}$ be a relation on a set $\mathcal{E} \subseteq \mathbb{E}$. We say that $\leq$ is $\mathcal{R}$ -smooth if for each $S \in \mathbb{S}$ holds $\min(S, \leq) \neq \emptyset$ whenever there is some $E \in \mathcal{E}$ such that $E \subseteq S$. We say that $\leq$ is K -minimal if $\min(\leq) = \{K\}$.

One might compare the notion of smoothness with the notion of a well-founded relation. The difference is that one demands the existence of minimal elements only for certain elements of interest, which is a more liberal requirement. The closest related notion is the notion of smoothness by Kraus, Lehmann and Magidor [24], for which it has been shown that one cannot replace smooth relations with well-founded relations [25]. K -minimality means that K is the (globally unique) minimal element of $\leq$.

For a $\mathcal{R}$ -smooth relation, we define a corresponding choice function for $\mathcal{R}$ and some K , which will act as a fallback value. Linear choice functions for $\mathcal{R}$ are then such choice functions for $\mathcal{R}$ given by a $\mathcal{R}$ -smooth relation that is a linear order on $\mathbb{E}$. If the relation is also K -minimal, we say that it is a K -minimal linear choice function for $\mathcal{R}$, i.e., when the fallback value K is encoded as the (globally) minimal element.

Definition 4.2 (linear choice function). Let $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$ be a restricted choice structure and let $K \in \mathbb{E}$. For each $\mathcal{R}$ -smooth linear order $\ll \subseteq \mathbb{E} \times \mathbb{E}$ on $\mathbb{E}$, we define the function $\nabla_{\ll}^K : \mathbb{S} \rightarrow \mathbb{E}$ as:

$$\nabla_{\ll}^K(S) = \begin{cases} E & \text{if } \min(S, \ll) = \{E\} \\ K & \text{otherwise} \end{cases} \quad (\star)$$

A function $\nabla : \mathbb{S} \rightarrow \mathbb{E}$ is a linear choice function for $\mathcal{R}$ if $\nabla = \nabla_{\ll}^K$ for some $\mathcal{R}$ -smooth linear order $\ll$. Additionally, we say that $\nabla_{\ll}^K$ is K -minimal if $\ll$ is also K -minimal.

Note that $\ll$ in Definition 4.2 is a linear order on the full set $\mathbb{E}$. In contrast, the relation $\leq$ in Definition 4.1 is a relation on any subset of $\mathbb{E}$. The additional flexibility in the latter case is required for the proofs of theorems in Section 5.

The following proposition witnesses that linear choice functions for a restricted choice structure $\mathcal{R}$ are indeed choice functions for $\mathcal{R}$. Moreover, when one assumes the Axiom of Choice, it is guaranteed that there exists some K -minimal linear choice function $\mathcal{R}$.

Proposition 4.3. Let $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$ be a restricted choice structure and let $K \in \mathbb{E}$. The following statements hold:

- (a) $\nabla_{\ll}^K : \mathbb{S} \rightarrow \mathbb{E}$ is a choice function for $\mathcal{R}$ with fallback K for any $\mathcal{R}$ -smooth linear order $\ll \subseteq \mathbb{E} \times \mathbb{E}$.
- (b) Assume the Axiom of Choice. There exists a K -minimal linear choice function for $\mathcal{R}$.

Proof. We start by showing Statement (a). If some $E \in \mathbb{E}$ with $E \subseteq S$ exists, then $\min(S, \ll) = \{E'\}$ is non-empty and thus, according to $(\star)$, we have $\nabla_{\ll}^K(S) = E'$. This implies that we have $\nabla_{\ll}^K(S) \subseteq S$. We obtain that $\nabla_{\ll}^K$ is a choice function for $\mathcal{R}$. Moreover, if no $E \in \mathbb{E}$ with $E \subseteq S$ exists, then we have $\nabla_{\ll}^K(S) = K$ due to $(\star)$. Consequently, K is the fallback of $\nabla_{\ll}^K$.

For Statement (b), assume the Axiom of Choice. It is known that the Axiom of Choice is equivalent to the well-ordering theorem [19], which states that on every set S there is a linear order $\leq \subseteq S \times S$ that is well-founded. Let $\ll' \subseteq \mathbb{E} \times \mathbb{E}$ be such a well-founded linear order on $\mathbb{E}$. Because $\ll'$ is a well-founded linear order and $\mathbb{E}$ is non-empty, there exist some unique minimal singleton set $\{K'\} = \min(\mathbb{E}, \ll')$. Now, let $\ll \subseteq \mathbb{E} \times \mathbb{E}$ be the linear order, in which K and K' are mutually substituted. This is the relation:

$$\begin{aligned} \ll = & (\ll' \setminus \{(K, S), (S, K), (K', S), (S, K') \mid S \in \mathbb{E}\}) \cup \{(K, K), (K', K')\} \\ & \cup \{(K', S) \mid K \ll' S \text{ and } S \neq K\} \cup \{(S, K') \mid S \ll' K \text{ and } S \neq K\} \\ & \cup \{(K, S) \mid K' \ll' S \text{ and } S \neq K'\} \cup \{(S, K) \mid S \ll' K' \text{ and } S \neq K'\} \\ & \cup \{(K', K) \mid K' \ll' K\} \cup \{(K, K') \mid K \ll' K'\} \end{aligned}$$

One can show that the relation $\ll$ is a well-founded linear order on $\mathbb{E}$. Clearly, we have $\min(\mathbb{E}, \ll) = \{K\}$ and thus, $\ll$ is K -minimal, and because $\ll$ is well-founded, $\ll$ is $\mathcal{R}$ -smooth. Statement (b) follows by employing the latter observation and Statement (a). $\square$

Example 4.4 (Continued from Example 1.3). We consider the restricted choice structure $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$, where A , $\mathbb{S}$ and $\mathbb{E}$ are from Examples 1.1–1.3, and let $K = \{\text{pretzels}, \text{nachos}, \text{dips}, \text{chillies}\}$. We let $\ll \subseteq \mathbb{E} \times \mathbb{E}$ be the following linear order on $\mathbb{E}$:

$$\begin{aligned} K &\ll \{\text{nachos}, \text{dips}, \text{chillies}\} \ll \{\text{pretzels}, \text{chocolate}\} \ll \{\text{pretzels}\} \\ &\ll \{\text{chocolate}\} \ll \{\text{chocolate}, \text{nachos}, \text{dips}, \text{chillies}\} \ll A \end{aligned}$$

One can see that $\ll$ is K -minimal and $\mathcal{R}$ -smooth. By employing $\ll$, we obtain the function $\nabla_{\text{Ex}} = \nabla_{\ll}^K$. According to Proposition 4.3, ∇_{Ex} is a K -minimal linear choice function for $\mathcal{R}$. For $S = \{\text{nachos}, \text{pretzels}, \text{dips}\}$ from Example 1.1 we obtain $\nabla_{\text{Ex}}(S) = \{\text{pretzels}\}$; and for $S' = \{\text{nachos}, \text{dips}\}$ from Example 4.4, we obtain the fallback value $\nabla_{\text{Ex}}(S') = K$. ■

Proposition 4.3 guarantees existence of K -minimal linear choice function for every restricted choice structure $\mathcal{R}$. In the following section, we will axiomatize such choice functions.

5 Axiomatics of K -minimal Linear Choice Functions

We axiomatise K -minimal linear choice functions for union-closed and arbitrary restricted choice structures. Given a restricted choice structure $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$, we make use of the following postulates for some fixed $K \in \mathbb{E}$:

- (SS0) If $\mathbb{E} \cap \mathcal{P}(S) \neq \emptyset$, then $\nabla(S) \subseteq S$.
- (SS1) If $\nabla(S) \not\subseteq S$, then $\nabla(S) = K$.
- (SS2) If $K \subseteq S$, then $\nabla(S) = K$.
- (SS3) If $\nabla(S_1) \subseteq S_2$ and $\nabla(S_2) \subseteq S_1$, then $\nabla(S_1) = \nabla(S_2)$.
- (SS4) If $\nabla(S_1) \subseteq S_1$ and $S_1 \subseteq S_2$, then $\nabla(S_2) \subseteq S_2$.
- (SS5) If $\nabla(S_i \cup S_{i+1}) = S_i$ for $0 \leq i \leq n$, then $S_0 \neq S_n$ implies $\nabla(S_0 \cup S_n) \neq S_n$.
- (SS6) If $\nabla(S_1 \cup S_2) = S_3$, then $\nabla(S_1 \cup S_3) = S_3$.

By (SS0), we ensure that a choice is made among the elements of S whenever $\mathbb{E}$ permits this. The postulate (SS1) describes that either a choice is made among S or the function falls back to K . With (SS2), we express that when K is a subset of S , we must choose exactly K . The postulate (SS3) demands that if S_1 and S_2 are mutually supersets of the choices among them, then choosing among S_1 or S_2 leads to the same result. With (SS4) we obtain that choosing among elements of S is inherited to all super-sets of S . With (SS5) one prevents cyclic situations among the potential choices. The postulate (SS6) describes that when chooses S_3 from S , then for each subset $S' \subseteq S$, we have that S_3 is chosen from $S' \cup S_3$, i.e., the elements of S_3 are prevalent against the other elements of S .

The first main theorem of this paper is that (SS0)–(SS6) exactly characterizes K -minimal linear choice functions for union-closed restricted choice structures.

Because of the limited space, we present an outline of the proof of the following Theorem 5.1. The full proof is given in the supplemental material⁵.

Theorem 5.1. *Assume the Axiom of Choice. Let $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$ be a union-closed restricted choice structure and let $K \in \mathbb{E}$. A function $\nabla : \mathbb{S} \rightarrow \mathbb{E}$ is a K -minimal linear choice function for $\mathcal{R}$ if and only if the axioms (SS0)–(SS6) are satisfied⁷*

Proof (outline). [“ $\Rightarrow$ ”] The left-to-right direction amounts to checking the satisfaction of the postulates point-by-point. [“ $\Leftarrow$ ”] The right-to-left direction consists of three steps. In *Step 1*, one constructs a relation $\leq$ on an appropriate subset $\mathcal{E} \subseteq \mathbb{E}$ by the following encoding scheme [33]: $A_1 \leq A_2$ if $\nabla(A_1 \cup A_2) = A_1$. The relation $\leq$ is reflexive, antisymmetric, consistent and $\mathcal{R}$ -smooth on $\mathcal{E}$. However, $\leq$ is not a linear order on $\mathbb{E}$. In *Step 2*, Suzumura’s theorem [35] yields an extension of the relation $\leq$ to a $\mathcal{R}$ -smooth linear order $\ll$ on $\mathcal{E}$. *Step 3* consists of expanding $\ll$ to a linear order $\ll$ on $\mathbb{E}$ such that $\nabla = \nabla_K^{\ll}$. $\square$

The second main result is an axiomatization of K -minimal linear choice functions for arbitrary restricted choice structure, by axioms that follow the same ideas as (SS1)–(SS6).

Theorem 5.2 (Representation Theorem). *Assume the Axiom of Choice. Let $\mathcal{R} = \langle A, \mathbb{S}, \mathbb{E} \rangle$ be a restricted choice structure and let $K \in \mathbb{E}$. A function $\nabla : \mathbb{S} \rightarrow \mathbb{E}$ is a K -minimal linear choice function for $\mathcal{R}$ if and only if the axioms (SS0)–(SS4) and the following axioms are satisfied:*

- (SS5E) *If $S_i \cup S_{i+1} \subseteq S_{i,i+1}$ and $\nabla(S_{i,i+1}) = S_i$ for $0 \leq i \leq n$, then $S_n \cup S_0 \subseteq S_{n,0}$ and $S_n \neq S_0$ imply $\nabla(S_{n,0}) \neq S_n$.*
- (SS6E) *If $S_{1,2} = S_1 \cup S_2$ and $S_{1,3} = S_1 \cup S_3$ and $\nabla(S_{1,2}) = S_3$, then $\nabla(S_{1,3}) = S_3$.*

The proof of Theorem 5.2 is given in the supplemental material⁵. The postulates (SS5E) and (SS6E) are variations of (SS5) and (SS6) that take into account that the union of elements of $\mathbb{S}$ might not be in $\mathbb{S}$. Note that one demands $S_i \cup S_{i+1} \subseteq S_{i,i+1}$ in (SS5E) and not $S_i \cup S_{i+1} = S_{i,i+1}$.

6 Applications in Knowledge Representation

In this section, we discuss instantiations of union-closed restricted choice structures and the respective interpretation of linear choice functions and Theorem 5.1 in the context of these applications.

⁷ As usual, satisfaction of (SS0)–(SS6) (by ∇) means that the properties described by (SS0)–(SS6) hold for all $S, S_0, S_1, \dots \in \mathbb{S}$ for the specifically considered ∇ . Note that K and $\mathbb{E}$ in (SS0)–(SS6) are externally given, and thus, are not all-quantified.

6.1 Linear Choice in Theory Change

In theory change, one asks how to change a knowledge base K according to new information S . Specifically for revision, one wants that S holds after revising K . Notably, in the framework by Katsuno and Mendelzon [20] a change operator is a function $\mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$, where $\mathcal{L}$ is a language of propositional logic. We adapt the notion of a change operator to the setting here. When $\mathbb{S} \subseteq \mathcal{P}(A)$ is a system of subsets of some set A , we define a change operator as a function $\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$. The following definition defines a change operator that is based on a choice function.

Definition 6.1. *A change operator $\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$ for $\mathbb{S}$ (over A) is called choice-based if for each $K \in \mathbb{S}$ there is a choice function C_K for some restricted choice structure $\mathcal{R}_K = \langle A, \mathbb{S}, \mathbb{E}_K \rangle$ with $K \in \mathbb{E}_K$ such that:*

$$K \odot S = C_K(S)$$

We say $\odot$ fits the $\mathbb{S}$ -indexed family of restricted choice structure $\{\langle A, \mathbb{S}, \mathbb{E}_K \rangle\}_{K \in \mathbb{S}}$. If each C_K is a K -minimal linear choice function for $\mathcal{R}$, we say that $\odot$ is linear.

Choice functions have already been employed to define change operators [27, 18]. The novelty here is the restriction on the output side; leading to a model of change operators for agents that are *not able* to conduct certain changes. In such a setting AGM revision operators are not realizable in general [32] or learnable [3]. In the following theorem, we employ Theorem 5.1 to give an axiomatization for linear choice-based change operators.

Theorem 6.2. *Assume the Axiom of Choice and let $\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$ be a change operator for a union-closed set of sets $\mathbb{S} \subseteq \mathcal{P}(A)$. The operator $\odot$ is linear choice-based if and only if the following postulates are satisfied:*

- (LCR1) $K \odot S \subseteq S$ or $K \odot S = K$.
- (LCR2) If $K \subseteq S$, then $K \odot S = K$.
- (LCR3) If $K \odot S_1 \subseteq S_2$ and $K \odot S_2 \subseteq S_1$, then $K \odot S_1 = K \odot S_2$.
- (LCR4) If $K \odot S_1 \subseteq S_1$ and $S_1 \subseteq S_2$, then $K \odot S_2 \subseteq S_2$.
- (LCR5) If $K \odot (S_i \cup S_{i+1}) = S_i$ for $0 \leq i \leq n$,
then $S_0 \neq S_n$ implies $K \odot (S_0 \cup S_n) \neq S_n$.
- (LCR6) If $K \odot (S_1 \cup S_2) = S_3$, then $K \odot (S_1 \cup S_3) = S_3$.

Proof (sketch). We show each direction independently. [“ $\Rightarrow$ ”] Let $\odot$ be a linear choice-based change operator. Inspecting Definition 6.1 reveals that for every K there is a K -minimal linear choice function C_K for $\mathcal{R}_K$ such that $K \odot S = C_K(S)$. By employing Theorem 5.1 we obtain that C_K satisfies (SS0)–(SS6) for each K . From the latter, one obtains easily that (LCR1)–(LCR6) are satisfied, by substituting $\nabla(S)$ by “ $K \odot S$ ”. [“ $\Leftarrow$ ”] Assume that $\odot$ satisfies (LCR1)–(LCR6). For each fixed K , define a function $C_K : \mathbb{S} \rightarrow \mathbb{E}_K$ with $K \odot S = C_K(S)$ and

$\mathbb{E}_K = \bigcup_{S \in \mathbb{S}} \{K \odot S\}$. One can see easily that each C_k satisfies (SS1)–(SS6) by replacing “ $K \odot S$ ” in every postulate (LCR1)–(LCR6) by “ $C_K(S)$ ”. The postulates (SS0) is satisfied because (SS1) is satisfied and because $\mathbb{E}_K$ contains only elements that in the image of $K \odot S$. By employing Theorem 5.1 we obtain that each C_K is a K -minimal linear choice function and, hence, that $\odot$ is linear. $\square$

The postulates (LCR1)–(LCR6) are reformulations of the postulates (SS1)–(SS6). Note that there is no counterpart of (SS0); inspecting Theorem 6.2 reveals that the definition of linear change operator is made in a way that the integrated choice function satisfies already (SS0). Note that Proposition 4.3 guarantees that linear choice-based change operators always exist.

Corollary 6.3. *Assume the Axiom of Choice and let $\mathbb{S} \subseteq \mathcal{P}(A)$ be a set of sets. For any $\mathbb{S}$ -indexed family of restricted choice structure $\{\langle A, \mathbb{S}, \mathbb{E}_K \rangle\}_{K \in \mathbb{S}}$ with $K \in \mathbb{E}_K$ for every $K \in \mathbb{S}$ there is a linear choice-based operator that fits that family.*

We consider the postulates (LCR1)–(LCR6). In theory change the postulates (LCR1)–(LCR4) are known⁸ as (Relative Success), (Idempotence), (Right-Reciprocity), and (Success Monotonicity). (LCR1) is especially known from non-prioritized revision [16]. In non-prioritized revision, one provides change operators for agents that are not willing to accept every new information unquestioned and fully [13,17,23,29]. However, the kind of changes we are drafting here differ from non-prioritized change conceptually. It is not that an agent is not willing to accept some new information, but she cannot because of the restricted outputs. Technically speaking, in non-prioritized change, the restriction is on how to deal with the inputs within the language. Restricted choice is about restrictions on the meta-level (especially on the outputs). Thus, (LCR1) seems to be a postulate that is not exclusive to non-prioritized change; it is more of an expression of dealing with restrictions through a fallback value. The following example reinterprets our running example in the context of change, demonstrating both cases (LCR1) describes.

Example 6.4 (Continued from Example 4.4). We consider $K = \{\text{pretzels}, \text{nachos}, \text{dips}, \text{chillies}\}$ from Example 4.4. In the context of change, K stands for the initial information. The linear order $\ll$ from Example 4.4 gives rise to ∇_{Ex} , which we employ for changing K , by setting $K \odot S = \nabla_{\text{Ex}}(S)$. We obtain, e.g., $K \odot \{\text{nachos}, \text{pretzels}, \text{dips}\} = \{\text{pretzels}\}$ and $K \odot \{\text{nachos}, \text{dips}\} = K$. $\blacksquare$

The postulate (LCR2) describes that when the initial information K is a suitable choice, the operator $\odot$ has to output K . This conforms with the special role the initial beliefs have in many belief change approaches [12]. In a certain way, (LCR2) is a basic form of “minimal change”. Note that, minimal change in AGM [15,1] involves that when $K \cap S \neq \emptyset$, then $K \odot S = K \cap S$. However, in the restricted choice setting this is not always possible, as $K \cap S \notin \mathbb{E}$ might hold. The remaining postulates (LCR3)–(LCR6) deal with the nature of linear orders

⁸ When one reads $\subseteq$ as some kind of entailment.

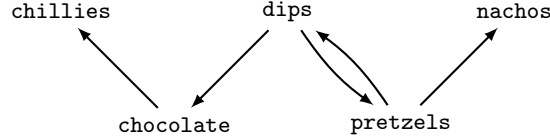


Fig. 1: Illustration of the abstract argumentation framework from Example 6.7.

over sets. Note that the postulates (LCR5) and (LCR6) seem to be novel for the change context; they are different from those for linear orders on alternatives [6].

6.2 Linear Choice in Argumentation

Formal argumentation [2] deals with conflicting pieces of information, where conflicting information is modelled as *arguments* in a discussion. Dung [11] proposed *abstract argumentation frameworks* (AFs) as a directed graph, where arguments are vertices and an *attack* between two arguments is a directed edge. Formally, given a (possible infinite) set of arguments $\mathcal{U}$, an *abstract argumentation framework* (AF) for $\mathcal{U}$ is a directed graph $F = \langle A, R \rangle$, where $A \subseteq \mathcal{U}$ is a (often finite) non-empty set of arguments and R is an *attack relation* $R \subseteq A \times A$. With $\text{AF}[\mathcal{U}]$ we denote the set of all abstract argumentation frameworks for $\mathcal{U}$.

There are different approaches to the semantics of abstract argumentation frameworks [4]. Most prominently, extension-based semantics assign to each AF $F = \langle A, R \rangle$ a set of sets of arguments $\sigma(F) \subseteq \mathcal{P}(A)$ (called the σ -extensions of F). The intended meaning is that $\sigma(F)$ represents the viable solutions of the conflict resolution between the arguments in F . A generalization of extension-based semantics are extension-ranking semantics, which additionally equip the extensions with an ordering [34], i.e., functions τ that map each AF to an ordering $\leq_F^\tau$ over some subsets of $\mathcal{P}(A)$. Extension-ranking semantics provide much more structure to the solutions given by extension-based semantics, i.e., that some extensions are “better” than others. We define a semantics for AFs that is based on choice functions.

Definition 6.5. A choice-based extension semantics for $\mathcal{U}$ is a function Π that maps each AF $F = \langle A, R \rangle$ for $\mathcal{U}$ to a choice function $\Pi_F : \mathcal{P}(A) \rightarrow \mathbb{E}_F$ for some choice structure $\mathcal{R}_F = \langle A, \mathcal{P}(A), \mathbb{E}_F \rangle$ where $\mathbb{E}_F$ is non-empty. We say that Π is linear, if every Π_F is a K_F -minimal linear choice function for some $K_F \in \mathbb{E}_F$.

Choice-based extension semantics provide for every argumentation framework F a restricted choice function Π_F over the arguments of F . A instantiation for $\mathbb{E}_F$ could be the extensions of F (with respect to some extension-based semantics). Given that, one can interpret a choice $\Pi_F(E)$ for some $E \subseteq A$ as answer to the question of “which arguments in E (if any) are conflict free?”. Some AFs can have a big number of arguments, hence it is of interest to only look at a selection of extensions that only concern these arguments. This is also known extension selection [22, 7, 34].

Next, we consider the axiomatic linear choice-based extension semantics provided by Theorem 5.1.

Theorem 6.6. *Assume the Axiom of Choice and let Π be a choice-based extension semantics for $\mathcal{U}$. The semantics Π is linear if and only if the following postulates are satisfied (for a suitable K_F for each $F \in \text{AF}[\mathcal{U}]$):*

- (LCA1) $\Pi_F(E) \subseteq E$ or $\Pi_F(E) = K_F$.
- (LCA2) If $K_F \subseteq \Pi_F(E)$, then $\Pi_F(E) = K_F$.
- (LCA3) If $\Pi_F(E_1) \subseteq E_2$ and $\Pi_F(E_2) \subseteq E_1$, then $\Pi_F(E_1) = \Pi_F(E_2)$.
- (LCA4) If $\Pi_F(E_1) \subseteq E_1$ and $E_1 \subseteq E_2$, then $\Pi_F(E_2) \subseteq E_2$.
- (LCA5) If $\Pi_F(E_i \cup E_{i+1}) = E_i$ for $0 \leq i \leq n$,
then $E_0 \neq E_n$ implies $\Pi_F(E_0 \cup E_n) \neq E_n$.
- (LCA6) If $\Pi_F(E_1 \cup E_2) = E_3$, then $\Pi_F(E_1 \cup E_3) = E_3$.

Proof. We show each direction independently. [“ $\Rightarrow$ ”] Let Π be a linear choice-based extension semantics. Then for every $F \in \text{AF}[\mathcal{U}]$ there is a K_F -minimal linear choice function Π_F for R_F . By employing Theorem 5.1, we obtain that Π_F satisfies (SS1)–(SS6) for each K_F , which yields the satisfaction of (LCA1)–(LCA6). [“ $\Leftarrow$ ”] Assume that Π satisfies (LCA1)–(LCA6). This implies that every function Π_F satisfies (SS1)–(SS6). Now define $\mathbb{E}_F = \bigcup_{E \in \mathcal{P}(A)} \{\Pi_F(E)\}$. Then, the satisfaction of (SS0) by Π_F is guaranteed by satisfaction of (SS1) and by the definition of $\mathbb{E}_F$. Because Π_F satisfies (SS0)–(SS6), we obtain that Π_F is a K_F -minimal linear choice function from Theorem 5.1, and thus, that Π is linear. $\square$

A direct connection to principles for abstract argumentation semantics [5,36] is not obvious. This is because, from the perspective of argumentation, much of the quality of the results $\Pi_F(E)$ depends on the internally chosen set $\mathbb{E}_F$. However, for the discussion here, we just assume that each element in $\mathbb{E}_F$ is well-behaving, e.g., conflict free, i.e., for $E \in \mathbb{E}_F$ we have $a, b \in E$ implies that $(a, b) \notin R$ [11]. Hence, under this assumption, our restriction of $\Pi_F(E)$ to $\mathbb{E}_F$ guarantees well-behaving outputs.

Example 6.7 (Continued from Example 6.4). We interpret Examples 1.1–4.4 in the context of argumentation. Here, the elements of A are understood as arguments and the elements of $\mathbb{E}$ are meant to be the “feasible” subset of arguments. The linear order $\ll$ provides an ordering on $\mathbb{E}$, where the smaller elements are more “feasible”. An argumentation framework that suits this scenario is the framework $F = \langle A, R \rangle$ with $R = \{(\text{pretzels}, \text{nachos}), (\text{pretzels}, \text{dips}), (\text{dips}, \text{pretzels}), (\text{dips}, \text{chocolate}), (\text{chocolate}, \text{chillies})\}$. For an illustration, see Figure 1. Example of outputs of the choice-based extension semantics Π are $\Pi_F(\{\text{nachos}, \text{pretzels}, \text{dips}\}) = \{\text{pretzels}\}$ and $\Pi_F(\{\text{nachos}, \text{dips}\}) = K$. $\blacksquare$

Choice-based extension semantics seem to be an interesting novel type of abstract argumentation semantics. However, how linear choice-based extension

semantics handles fallback values does not seem appropriate. In Example 6.7, we observe the behaviour of $\Pi_F(\{\mathbf{nachos}, \mathbf{dips}\}) = K$ in Example 6.7. This is because the fallback value is the minimal element of the respective linear order. Thus, one might not want the fallback value to be the minimal element in abstract argumentation. For instance, for some semantics, the empty extension $\emptyset$ would be a suitable fallback value. However, when $K_F = \emptyset$ is the minimal element, then $\Pi_F(E) = \emptyset$ holds for all $E \in \mathbb{S}$.

7 Conclusion

In this paper, we considered the novel setting of restricted choices. A full axiomatization (Theorem 5.2) is given for K -minimal linear choice functions, i.e., those choice functions that can be represented by a linear order over sets of sets where a fallback value is encoded as the minimal element. As given by Proposition 4.3, such choice functions always exist regardless of which restrictions on choices are given. We showed that Theorem 5.2 can be simplified to Theorem 5.1 for choice structures where the input is closed under union, i.e., in this setting, (SS1)–(SS6) fully characterise K -minimal linear choice functions. We give a first hint on the applications of linear choice function in theory change and argumentation.

In future work, we will transfer the approach of this paper to the area of non-monotonic reasoning, especially preferential reasoning [24]. It turns out that transferring preferential reasoning to other domains is axiomatically challenging [30]. As non-monotonic reasoning and belief change are known to be connected [14], we are optimistic that our approach here might have applications in the axiomatization of non-monotonic reasoning approaches. Furthermore, for the area of argumentation, the idea of choice-based semantics sketched here is a promising and interesting area to explore.

Another avenue of future work is further exploration of the framework itself. We will consider alternative fallback behaviours for choice functions and explore their conceptual relevance. For that, identifying natural fallback behaviours for different applications in knowledge representation and beyond will be very helpful.

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References

1. Alchourrón, C.E., Gärdenfors, P., Makinson, D.: On the logic of theory change: Partial meet contraction and revision functions. *The Journal of Symbolic Logic* **50**(2), 510–530 (1985)
2. Atkinson, K., Baroni, P., Giacomin, M., Hunter, A., Prakken, H., Reed, C., Simari, G.R., Thimm, M., Villata, S.: Towards artificial argumentation. *AI Mag.* **38**(3), 25–36 (2017). <https://doi.org/10.1609/AIMAG.V38I3.2704>
3. Baltag, A., Gierasimczuk, N., Smets, S.: Truth-tracking by belief revision. *Studia Logica* **107**(5), 917–947 (2019). <https://doi.org/10.1007/s11225-018-9812-x>
4. Baroni, P., Gabbay, D., Giacomin, M., Van der Torre, L.: Handbook of formal argumentation (2018)
5. Baroni, P., Giacomin, M.: On principle-based evaluation of extension-based argumentation semantics. *Artificial Intelligence* **171**(10-15), 675–700 (2007). <https://doi.org/10.1016/J.ARTINT.2007.04.004>
6. Belahcène, K., Gaigne, J., Lagrue, S.: On decisive revision operators and totally ordered information. In: Destercke, S., Martinez, M.V., Sanfilippo, G. (eds.) *Proceedings of the 16th International Conference on Scalable Uncertainty Management (SUM 2024)*. LNCS, vol. 15350, pp. 45–52. Springer (2024). https://doi.org/10.1007/978-3-031-76235-2_4
7. Bonzon, E., Delobelle, J., Konieczny, S., Maudet, N.: Combining extension-based semantics and ranking-based semantics for abstract argumentation. In: Thielscher, M., Toni, F., Wolter, F. (eds.) *Proceedings of the 16th International Conference, Principles of Knowledge Representation and Reasoning (KR 2018)*. pp. 118–127. AAAI Press (2018)
8. Bossert, W., Suzumura, K.: Consistency, choice, and rationality. Harvard University Press (2010)
9. Creignou, N., Papini, O., Pichler, R., Woltran, S.: Belief revision within fragments of propositional logic. *Journal of Computer and System Sciences* **80**(2), 427–449 (2014). <https://doi.org/10.1016/J.JCSS.2013.08.002>
10. Delgrande, J.P., Peppas, P.: Belief revision in Horn theories. *Artificial Intelligence* **218**, 1–22 (2015). <https://doi.org/10.1016/j.artint.2014.08.006>
11. Dung, P.M.: On the acceptability of arguments and its fundamental role in non-monotonic reasoning, logic programming and n-person games. *Artificial Intelligence* **77**(2), 321–358 (1995). [https://doi.org/10.1016/0004-3702\(94\)00041-X](https://doi.org/10.1016/0004-3702(94)00041-X)
12. Fermé, E.L., Hansson, S.O.: Belief Change - Introduction and Overview. *Springer Briefs in Intelligent Systems*, Springer (2018). <https://doi.org/10.1007/978-3-319-60535-7>
13. Fermé, E.L., Hansson, S.O.: Selective revision. *Studia Logica* **63**(3), 331–342 (1999). <https://doi.org/10.1023/A:1005294718935>
14. Gärdenfors, P., Makinson, D.: Nonmonotonic inference based on expectations. *Artificial Intelligence* **65**(2), 197–245 (1994). [https://doi.org/10.1016/0004-3702\(94\)90017-5](https://doi.org/10.1016/0004-3702(94)90017-5)
15. Gärdenfors, P.: Knowledge in flux : modeling the dynamics of epistemic states. MIT Press Cambridge (1988)
16. Hansson, S.O.: A survey of non-prioritized belief revision. *Erkenntnis* **50**(2), 413–427 (1999). <https://doi.org/10.1023/A:1005534223776>
17. Hansson, S.O., Fermé, E.L., Cantwell, J., Falappa, M.A.: Credibility limited revision. *The Journal of Symbolic Logic* **66**(4), 1581–1596 (2001)

18. Haret, A.: Choosing what to believe: belief change through the lens of rational choice (2020). <https://doi.org/10.34726/hss.2020.42785>
19. Jech, T.J.: Set Theory. Springer Monographs in Mathematics, Springer, Berlin, Germany, 3 edn. (2002). <https://doi.org/10.1007/3-540-44761-X>
20. Katsuno, H., Mendelzon, A.O.: Propositional knowledge base revision and minimal change. *Artificial Intelligence* **52**(3), 263–294 (1991). [https://doi.org/10.1016/0004-3702\(91\)90069-V](https://doi.org/10.1016/0004-3702(91)90069-V)
21. Kelly, J.S.: Social choice theory: An introduction. Springer Science & Business Media (2013)
22. Konieczny, S., Marquis, P., Vesic, S.: On supported inference and extension selection in abstract argumentation frameworks. In: Destercke, S., Denoeux, T. (eds.) *Proceedings of the 13th European Conference Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2015)*. LNCS, vol. 9161, pp. 49–59. Springer (2015). https://doi.org/10.1007/978-3-319-20807-7_5
23. Konieczny, S., Pino Pérez, R.: Improvement operators. In: Brewka, G., Lang, J. (eds.) *Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR 2008)*. pp. 177–187. AAAI Press (2008)
24. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence* **44**(1-2), 167–207 (1990). [https://doi.org/10.1016/0004-3702\(90\)90101-5](https://doi.org/10.1016/0004-3702(90)90101-5)
25. Lehmann, D., Magidor, M.: What does a conditional knowledge base entail? *Artificial Intelligence* **55**(1), 1–60 (1992). [https://doi.org/10.1016/0004-3702\(92\)90041-U](https://doi.org/10.1016/0004-3702(92)90041-U)
26. Makinson, D.: Five faces of minimality. *Studia Logica* **52**(3), 339–379 (1993). <https://doi.org/10.1007/BF01057652>
27. Rott, H.: Change, Choice and Inference: A Study of Belief Revision and Nonmonotonic Reasoning. Oxford University Press, New York (2001)
28. Sauerwald, K.: Credibility-limited revision for epistemic spaces. In: Sanfilippo, G., Destercke, S., Martinez, V. (eds.) *Proceedings of the 16th International Conference on Scalable Uncertainty Management (SUM 2024)* (2024). <https://doi.org/10.24963/kr.2022/42>
29. Sauerwald, K., Beierle, C.: Decrement operators in belief change. In: Kern-Isberner, G., Ognjanovic, Z. (eds.) *Proceedings of the 15th European Conference Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2019)*. LNCS, vol. 11726, pp. 251–262. Springer (2019). https://doi.org/10.1007/978-3-030-29765-7_21
30. Sauerwald, K., Kontinen, J.: A first peek into preferential logics with team semantics. In: Gierasimczuk, N., Heyninck, J. (eds.) *Proceedings of the 22nd International Workshop on Nonmonotonic Reasoning (NMR 2024) co-located with 21st International Conference on Principles of Knowledge Representation and Reasoning (KR 2024)*, Hanoi, Vietnam, November 2-4, 2024. *CEUR Workshop Proceedings*, vol. 3835, pp. 94–100. CEUR-WS.org (2024)
31. Sauerwald, K., Skiba, K., Fermé, E., Meyer, T.: Axiomatics of restricted choices by linear orders of sets with minimum as fallback. *CoRR* **abs/2506.03315** (2025). <https://doi.org/10.48550/ARXIV.2506.03315>
32. Sauerwald, K., Thimm, M.: The realizability of revision and contraction operators in epistemic spaces. In: Marquis, P., Ortiz, M., Pagnucco, M. (eds.) *Proceedings of the 21st International Conference on Principles of Knowledge Representation and Reasoning (KR 2024)*. pp. 665–670 (2024). <https://doi.org/10.24963/KR.2024/62>

33. Sen, A.K.: Choice functions and revealed preference. *The Review of Economic Studies* **38**(3), 307–317 (1971). <https://doi.org/10.2307/2296384>
34. Skiba, K., Rienstra, T., Thimm, M., Heyninck, J., Kern-Isberner, G.: Ranking extensions in abstract argumentation. In: Zhou, Z. (ed.) *Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI 2021)*. pp. 2047–2053. *ijcai.org* (2021). <https://doi.org/10.24963/IJCAI.2021/282>
35. Suzumura, K.: Remarks on the theory of collective choice. *Economica* **43**(172), 381–390 (1976). <https://doi.org/10.2307/2553273>
36. van der Torre, L., Vesic, S.: The principle-based approach to abstract argumentation semantics. *Journal of Logics and their Applications* **4**(8), 2735–2778 (2017)
37. Zermelo, E.: Beweis, daß jede menge wohlgeordnet werden kann. *Mathematische Annalen* **59**(4), 514–516 (1904). <https://doi.org/10.1007/BF01445300>